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EQUILIBRIA ON L-RETRACTS IN RIEMANNIAN MANIFOLDS

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Abstract. In this paper we introduce a class of subsets of Riemannian manifolds called L-retract. Next we consider a topological degree for set-valued upper semicontinuous maps defined on open sets of compact L-retracts in Riemannian manifolds. Then, we present a theorem on the existence of equilibria (or zeros) of an upper semicontinuous set-valued map with nonempty closed convex values satisfying the tangency condition defined on a compact L-retract in a Riemannian manifold.

1. Introduction

Let M be a Banach space and ϕ be a set-(or single) valued map from M into the family of nonempty closed subsets of M and let $S \subset M$. The existence of a solution to the set-valued constrained equation $0 \in \phi(x)$, $x \in S$, plays an important role in nonlinear analysis. A point $x \in S$ such that $0 \in \phi(x)$ is called "equilibrium" which originates from the calculus of variations and control problems. Ky Fan and F. Browder proved that given a compact convex set S in a Banach space M, an upper semicontinuous set-valued map $\phi : S \rightrightarrows M$ with closed convex values has an equilibrium provided it is inward (or tangent) in the sense that, for each $x \in S, \phi(x) \cap T_S(x) \neq \emptyset$ where $T_S(x)$ stands for the tangent cone to S at $x \in S$ defined in the sense of convex analysis; see [4, 5, 12].

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This result has been generalized in several directions by many authors; see, e.g. [7, 8, 18]. In [3] the authors proved that if $S \subset M$ is a compact L- retract with the nontrivial Euler characteristic $\chi(S) \neq 0$ and if $\phi : S \rightrightarrows M$ is an upper semicontinuous set-valued map with closed convex values satisfying the inwardness condition, then ϕ has an equilibrium. Here the inwardness condition means

$$\forall x \in S,\ \phi(x) \cap T_S(x) \neq \emptyset,$$

where $T_S(x)$ stands for the Clarke tangent cone to S at $x \in S$. If M is a smooth manifold and TM is its tangent bundle, then the existence of equilibria of a set- (or single) valued map $\phi : S \rightrightarrows TM$ such that $\phi(x) \subset T_xM$ may also be studied. In [16] we introduced a notion of Euler characteristic of an epi-Lipschitz subset S of a complete Riemannian manifold M and proved some equilibria theorems for this class of sets. We defined the Euler characteristic of S by using the Cellina-Lasota degree of upper semicontinuous mappings with compact convex values. In this paper we introduce a notion of L-retract in the setting of Riemannian manifolds. We assume that S is an L-retract in a Riemannian manifold M, therefore S is an absolute neighborhood retract. Then, a topological degree for a set-valued upper semicontinuous map $\Phi : \Omega \rightrightarrows TM$, where TM is the tangent bundle of M and Ω is an open set in a compact L-retract S, is presented. The presented topological degree also can be exploited to prove the existence of equilibria of an upper semicontinuous set-valued map with nonempty closed convex values satisfying the tangency condition defined on a compact L-retract with nontrivial Euler characteristic. These results are motivated by [3, 9, 10] and can be viewed as generalizations of the corresponding notions to the setting of manifolds.

2. Preliminaries

In this paper, we use the standard notations and known results of Riemannian manifolds, see, e.g. [11, 21]. Throughout this paper, M is a finite dimensional Riemannian manifold. As usual we denote by $B(x, \delta)$ the open ball centered at x with radius δ, by $\text{int}N(\text{cl}N)$ the interior (closure) of the set N. Also, let S be a nonempty closed subset of a Riemannian manifold M, we define $d_S : M \longrightarrow \mathbb{R}$ by

$$d_S(x) := \inf\{d(x, s) : s \in S\ \},$$

where d is the Riemannian distance on M. Moreover, $B(S, \varepsilon) := \{x \in M : d_S(x) < \varepsilon\}$. Recall that the set S in a Riemannian manifold M is called convex if every two points $p_1,\ p_2 \in S$ can be joined by a unique minimizing geodesic whose image belongs to S. For the point $x \in M$, $\exp_x : U_x \to M$ will stand for the exponential function at x, where U_x is an open subset of T_xM. Recall that $\exp_x$ maps straight lines of the tangent space T_xM passing through $0_x \in T_xM$

into geodesics of M passing through x. We will also use the parallel transport of vectors along geodesics. Recall that, for a given curve $\gamma : I \to M$, number $t_0 \in I$, and a vector $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = V_0$. Moreover, the map defined by $V_0 \mapsto V(t_1)$ is a linear isometry between the tangent spaces $T_{\gamma(t_0)}M$ and $T_{\gamma(t_1)}M$, for each $t_1 \in I$. In the case when γ is a minimizing geodesic and $\gamma(t_0) = x, \gamma(t_1) = y$, we will denote this map by L_{xy}, and we will call it the parallel transport from T_xM to T_yM along the curve γ. Note that, L_{xy} is well defined when the minimizing geodesic which connects x to y, is unique. For example, the parallel transport L_{xy} is well defined when x and y are contained in a convex neighborhood. In what follows, L_{xy} will be used wherever it is well defined.

REMARK 2.1. *Let M be a Riemannian manifold.*
(a) An easy consequence of the definition of the parallel translation along a curve as a solution to an ordinary linear differential equation, implies that the map

$$C : TM \to T_{x_0}M, \ C(x, \xi) = L_{xx_0}(\xi),$$

is continuous at (x_0, ξ_0), that is, if $(x_n, \xi_n) \to (x_0, \xi_0)$ in TM then $L_{x_n x_0}(\xi_n) \to L_{x_0 x_0}(\xi_0) = \xi_0$, for every $(x_0, \xi_0) \in TM$; see [1, Remark 6.11].
(b) By the continuity properties of the parallel transport and the geodesic, see [2, Theorem 35]*, for fixed point $z \in M$ and for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that:*

$$\|L_{xy}L_{zx} - L_{zy}\| \leq \varepsilon \qquad \text{provided that} \quad d(x, y) < \delta.$$

Recall that a real valued function f defined on Riemannian manifold M is said to satisfy a Lipschitz condition of rank k on a given subset S of M if $| f(x) - f(y) |\leq kd(x, y)$ for every $x, y \in S$, where d is the Riemannian distance on M. A function f is said to be Lipschitz near $x \in M$ if it satisfies the Lipschitz condition of some rank on an open neighborhood of x . A function f is said to be locally Lipschitz on M if f is Lipschitz near x, for every $x \in M$. Also, a set-valued map $F : X \rightrightarrows Y$, where $X,\ Y$ are topological spaces is said to be upper semicontinuous at x if for every open neighborhood U of $F(x)$, there exists an open neighborhood V of x, such that

$$y \in V \Longrightarrow F(y) \subseteq U.$$

Furthermore, a set-valued map $F : X \rightrightarrows Y$, where $X,\ Y$ are topological spaces, is said to be lower semicontinuous at x if for every open neighborhood U with $U \cap F(x) \neq \emptyset$, there exists an open neighborhood V of x, such that

$$y \in V \Longrightarrow F(y) \cap U \neq \emptyset.$$

It is worth pointing out that a set-valued map $F : X \rightrightarrows Y$, where X, Y are topological spaces, is said to be lower semicontinuous (upper semicontinuous) if F is lower semicontinuous (upper semicontinuous) at every point $x \in X$. Let us continue with the definition of the Clarke generalized directional derivative for locally Lipschitz functions on Riemannian manifolds; see [15, 17]. Suppose $f : M \to \mathbb{R}$ is a locally Lipschitz function on a Riemannian manifold M. Let $\phi_x : U_x \to T_xM$ be an exponential chart at x. Given another point $y \in U_x$, consider $\sigma_{y,v}(t) := \phi_y^{-1}(tw)$, a geodesic passing through y with derivative w, where (ϕ_y, y) is an exponential chart around y and $d(\phi_x \circ \phi_y^{-1})(0_y)(w) = v$. Then, the generalized directional derivative of f at $x \in M$ in the direction $v \in T_xM$, denoted by $f^\circ(x; v)$, is defined as

$$f^\circ(x, v) = \limsup_{y \to x,\ t \downarrow 0} \frac{f(\sigma_{y,v}(t)) - f(y)}{t}.$$

We recall some results of [15] which are needed in this paper.

Theorem 2.2. *Let M be a Riemannian manifold and $x \in M$. Suppose that the function $f : M \to \mathbb{R}$ is Lipschitz of rank K on an open neighborhood U of x. Then:*
(a) for each $y \in U$ the function $v \mapsto f^\circ(y; v)$ is finite, positive homogeneous, and sub-additive on T_yM, and satisfies

$$| f^\circ(y; v) | \leq K\|v\|.$$

(b) $f^\circ(y; v)$ is upper semicontinuous on $TM|_U$ and, as a function of v alone, is Lipschitz of rank K on T_yM, for each $y \in U$.
(c) $f^\circ(y; -v) = (-f)^\circ(y; v)$ for each $y \in U$ and $v \in T_yM$.

Let us present some definitions and properties of normal and tangent cones.

Definition 2.3. Let S be a nonempty closed subset of a Riemannian manifold M, $x \in S$ and (φ, U) be a chart of M at x. Then the (Clarke) tangent cone to S at x, denoted by $T_S(x)$ is defined as follows:

$$T_S(x) := d\varphi(x)^{-1}[T_{\varphi(S\cap U)}(\varphi(x))],$$

where $T_{\varphi(S\cap U)}(\varphi(x))$ is tangent cone to $\varphi(S \cap U)$ as a subset of $\mathbb{R}^n$ at $\varphi(x)$.

Obviously, $0_x \in T_S(x)$ and $T_S(x)$ is closed and convex.

Theorem 2.4. *Let S be a closed subset of a Riemannian manifold M, $x \in S$ and $v \in T_xM$. The following assertions hold.*
(i) If $d_S^\circ(x, v) = 0$, then $v \in T_S(x)$.
(ii) Conversely, if in addition M is complete and $v \in T_S(x)$, then $d_S^\circ(x, v) = 0$.

In the case of submanifolds of $\mathbb{R}^n$, the tangent space and the normal space are orthogonal to one another. In an analogous manner, for a closed subset S of a Riemannian manifold, the normal cone to S at x, denoted $N_S(x)$, is defined as the (negative) polar of the tangent cone $T_S(x)$, i.e.

$$N_S(x) := T_S(x)^\circ := \{\xi \in T_xM^* : \langle \xi, z\rangle \le 0 \quad \forall z \in T_S(x)\}.$$

3. Degree for set-valued tangent maps

In this section we first recall definitions of the Lefschetz number of a function $f : S \to S$ defined on an absolute neighborhood retract S, the Euler characteristic of S and the fixed point index of a compact continuous function defined on an open subset of S, where S is a subset of a metric space M. Then, we introduce a notion of L-retract in the setting of Riemannian manifolds and present a notion of topological degree for set-valued upper semicontinuous maps defined on open subsets of L-retracts. Finally, we prove a theorem on the existence of equilibria (or zeros) of an upper semicontinuous set-valued map with nonempty closed convex values satisfying the tangency condition defined on a compact L-retract in a Riemannian manifold.

A metric space S is an absolute neighborhood retract (ANR) if, given a metric space M, a closed subset $A \subset M$, and a continuous map $f : A \to S$, then f can be extended over some neighborhood of A in M. This is equivalent to say that, S is an ANR if, whenever S is a closed subset of a metric space M, then S is a neighborhood retract of M, i.e. there is an open subset U of M containing S and a continuous map $r : U \to S$ such that $r(x) = x$ for $x \in S$; see, e.g. [6, 22]. If S is a compact ANR, then it is homotopy dominated by a compact polyhedron and hence for any cohomology theory $H^*(.;\mathbb{Q})$ with rational coefficients, the graded vector space $H^*(S;\mathbb{Q})$ is of finite type. This means for all $q \ge 0$, $\beta^q := \dim_{\mathbb{Q}} H^q(S,\mathbb{Q}) < \infty$ and for almost all $q \ge 0, H^q(S,\mathbb{Q}) = 0$. Hence, for a given continuous map $f : S \to S$, one can consider the well-defined Lefschetz number $\lambda(f)$ as the following

$$\lambda(f) := \sum_{q\ge 0}(-1)^q \mathrm{tr} f^{*q} \in \mathbb{Q},$$

where $f^{*q} : H^q(S;\mathbb{Q}) \to H^q(S;\mathbb{Q})$ is the induced homomorphism and tr denotes the trace. The universal coefficient theorem (see [22]) yields $\lambda(f) \in \mathbb{Z}$ and the Euler characteristic of S is defined as

$$\chi(S) := \sum_{q\ge 0}(-1)^q \beta^q(S) = \lambda(i_S),$$

where i_S is the identity map on S; see [6, 22] for the details.

Assume that V is an open subset of S, $f : V \to S$ is a compact continuous function and its fixed point set is compact. Take an open set U_0 in a normed linear space E that r-dominates S; for the definition of r-dominates see [13]. Let

$s : S \to U_0$ and $r : U_0 \to S$ be such that $r \circ s = 1_S$. Then, the fixed point index of f denoted by $I(f, V)$ is defined by

$$I(f, V) := I(s \circ f \circ r, r^{-1}(V)),$$

where $I(s \circ f \circ r, r^{-1}(V))$ is the Leray-Schaulder index of $s \circ f \circ r$; see [13].

Assume that V is an open subset of S, $f : \text{cl}V \to S$ is compact and has all its fixed points in V. Moreover, assume that the fixed point set of f is compact. Then, the fixed point index $i(f, V)$ of f is given by

$$i(f, V) = I(f|_V, V).$$

Now we introduce the class of L-retracts in Riemannian manifolds.

DEFINITION 3.1. Let M be a Riemannian manifold and $S \subset M$. The set S is said to be an L-retract if there are a neighborhood U of S in M, a retraction $r : U \to S$ i.e. $(r(x) = x,\ x \in S)$, and a constant $L > 0$ such that

$$d(x, r(x)) \leq L d_S(x), \text{ for all } x \in U.$$

Therefore if M is a Riemannian manifold and S is an L-retract, then S equipped with a topology induced from M is an absolute neighborhood retract. Let us start with an example from [16]. Recall that a subset S of a Riemannian manifold M is said to be epi-Lipschitz if at every point $x \in S$, $N_S(x) \cap (-N_S(x)) = \{0\}$.

EXAMPLE 3.2. In [16] it is proved that any compact epi-Lipschitz subset of a complete Riemannian manifold is an L-retract. Indeed, it is proved that if S is a compact epi-Lipschitz subset of a complete Riemannian manifold M, then there exists a locally Lipschitz retraction for S, i.e. there are an open neighborhood U of S and a retraction $r : U \to S$ which is locally Lipschitz. Hence, for each $x \in U$, there exists $\epsilon(x) > 0$ such that the restriction of r to the open ball $B(x, 2\epsilon(x))$ is Lipschitz with constant $L(x) > 0$. By the compactness of S, there are $x_1, \ldots, x_k \in S$ such that $S \subseteq \bigcup_{i=1}^k B(x_i, \epsilon(x_i))$. Now set $L := \max\{L(x_i) :\ i = 1, \ldots, k\}$, and $V := \bigcup_{i=1}^k B(x_i, \epsilon(x_i))$. For each $x \in V$, the Hopf-Rinow theorem implies that there exists $y \in S$ such that $d_S(x) = d(y, x)$. Moreover, there exists $x_i \in S$ such that $x \in B(x_i, \epsilon(x_i))$ and $d(x, y) \leq d(x, x_i) < \epsilon(x_i)$. Hence $x,\ y \in B(x_i, 2\epsilon(x_i))$ and $d(r(x), r(y)) \leq L(x_i) d(x, y)$. Therefore

$$d(r(x), x) \leq d(x, y) + d(r(x), r(y)) \leq (L + 1) d(x, y).$$

Now we present a notion of topological degree for set-valued upper semicontinuous maps defined on open subsets of L-retracts. Suppose that Ω is a nonempty open set in a compact L-retact $S \subset M$ and $\Phi : \Omega \rightrightarrows TM$ is an upper semicontinuous map such that $\Phi(x) \subset T_x M$, for all $x \in \Omega$. The natural idea is to approximate Φ by a continuous tangent vector field, i.e. $f : \Omega \to TM$ such that $f(x) \in T_S(x)$, for all $x \in \Omega$. In the cases where $x \in S \mapsto T_S(x)$ is lower semicontinuous, for instance, in epi-Lipschitz subsets of Riemannian manifolds, by using Michael selection one can approximate Φ by a continuous vector field. However, in many other cases the lack of lower semicontinuity of $x \in S \mapsto T_S(x)$ makes it difficult to approximate Φ. Following [10], we overcome this difficulty by means of the next theorem.

THEOREM 3.3. *Let M be a complete Riemannian manifold and $\psi : M \rightrightarrows TM$ be a lower semicontinuous map with convex values such that for every $x \in M$, $\psi(x) \subset T_xM$, and $\Phi : M \rightrightarrows TM$ be an upper semicontinuous map with closed convex values such that for every $x \in M$, $\Phi(x) \subset T_xM$ and $\Phi(x) \cap \psi(x) \neq \emptyset$. Then for every small $\delta > 0$, there is a smooth vector field $f : M \to TM$ such that for every $x \in M$:*
(a) There exists $z_x \in \psi(x)$ such that $\|z_x - f(x)\| < \delta$.
(b) There exist $\bar{x} \in B(x, \delta)$ and $y_{\bar{x}} \in \Phi(\bar{x})$ such that $\|L_{\bar{x}x}(y_{\bar{x}}) - f(x)\| < \delta$.

PROOF. By Remark 2.1, the map $y \mapsto L_{yx}(\Phi(y))$ is upper semicontinuous in a neighborhood of x. Hence, the following set is open

$$U(x) := \{y \in B(x, \frac{r_x}{n}) : L_{yx}(\Phi(y)) \subseteq \Phi(x) + \frac{\delta}{2} B_{T_xM}\},$$

where r_x is the radius of a geodesic ball around x and $\frac{r_x}{n} \leq \delta$. Assume that $\mathbf{V} := \{V\}$ is an open star-refinement of the open cover $\mathbf{U} = \{U(x)\}_{x \in M}$. For any $x \in M$, choose $z_x \in \Phi(x) \cap \psi(x)$, and consider the open cover $\tau := \{T_V(x)\}_{V \in \mathbf{V}, x \in V}$ of M, where $T_V(x) := \{y \in V : \ L_{yx}(\psi(y)) \cap B(z_x, \frac{\delta}{2}) \neq \emptyset\}$. Note that $T_V(x)$ is open because of the lower semicontinuity of ψ. Let $\{\lambda_s\}_{s \in S}$ be a locally finite partition of unity subordinated to τ. Then for each $s \in S$ there exist $V_s \in \mathbf{V}$ and $x_s \in V_s$ such that $\lambda_s(x') = 0$, for $x' \notin T_{V_s}(x_s)$. We define the smooth vector field $f : M \to TM$ as follows,

$$f(x) = \sum_{s \in S} \lambda_s(x) L_{x_s x}(z_s), \ x \in M,$$

where $z_s = z_{x_s}$. Moreover, for each $x \in M$ and each s in the finite set $S(x) := \{s \in S : \ \lambda_s(x) \neq 0\}$ there exists $z'_s \in \psi(x)$ such that $\|L_{xx_s}(z'_s) - z_s\| < \delta$. Thus by the convexity of $\psi(x)$,

$$\sum_{s \in S(x)} \lambda_s(x) z'_s \in \psi(x).$$

Hence

$$\| \sum_{s \in S(x)} \lambda_s(x) z'_s - f(x)\| \leq \sum_{s \in S(x)} \lambda_s(x) \|L_{xx_s}(z'_s) - z_s\| < \delta,$$

and the proof of part (a) is complete.

In order to prove part (b), for each $x \in M$ and each $s \in S(x)$ we have $x \in T_{V_s}(x_s) \subset V_s$, where $x_s \in V_s$. Since $\mathbf{V}$ is a star-refinement of $\mathbf{U}$, there is $\bar{x} \in M$

such that $x, x_s \in U(\bar{x})$. Therefore $L_{x_s\bar{x}}(z_s) \in L_{x_s\bar{x}}(\Phi(x_s)) \subseteq \Phi(\bar{x}) + \frac{\delta}{2}B_{T_{\bar{x}}M}$ and $\bar{x} \in B(x, \delta)$. Set $M_x := \max_{s\in S(x)}\{\|z_s\|\}$. Then Remark 2.1 implies that

$$\|L_{x\bar{x}}L_{x_s x}(z_s) - L_{x_s\bar{x}}(z_s)\| < \frac{\delta}{2M_x}M_x = \frac{\delta}{2}.$$

The set $\Phi(\bar{x}) + \frac{\delta}{2}B_{T_{\bar{x}}M}$ is convex and

$$\sum_{s\in S(x)} \lambda_s(x)L_{x_s\bar{x}}(z_s) \in \Phi(\bar{x}) + \frac{\delta}{2}B_{T_{\bar{x}}M}.$$

Hence there is $y_{\bar{x}} \in \Phi(\bar{x})$ such that

$$\|\sum_{s\in S(x)} \lambda_s(x)L_{x_s\bar{x}}(z_s) - y_{\bar{x}}\| < \frac{\delta}{2}.$$

Therefore $\|L_{\bar{x}x}(y_{\bar{x}}) - f(x)\| < \delta$, as required.

Recall that the exponential map is defined on an open subset W of TM. Define a new map $F : W \to M \times M$ by $F(q, v) = (q, \exp_q(v))$. Along the same lines as [19, Lemma 5.12] since the topology on TM is generated by product open sets in local trivializations, one can deduce that for every $p \in M$, there exists a compact subset $U_{(p,0)} := \{(q, v) : \ q \in U_p, \|v\| \leq \delta_p\}$ of TM containing $(p, 0)$, where U_p is a compact neighborhood containing p, such that F is diffeomorphism on $U_{(p,0)}$. Hence there is $C_p > 0$ such that for every $(q, v), (q, w) \in U_{(p,0)}$,

$$d(\exp_q(v), \exp_q(w)) \leq C_p\|v - w\|.$$

Now let S be a compact subset of M. Then we may write $S \subseteq \cup_{i=1}^n U_{p_i},\ p_i \in S$ and set $\delta := \min\{\delta_{p_i},\ i = 1, ..., n\},\ C := \max\{C_{p_i} :\ i = 1, ..., n\}$. Hence for every $q \in S$, there is $p_i \in S$ such that $q \in U_{p_i}$. Thus for every $v, w \in B(0_q, \delta) \subseteq T_qM$,

$$d(\exp_q(v), \exp_q(w)) \leq C\|v - w\|.$$

Remark 3.4. Suppose that Ω is a nonempty open set in a compact L-retact $S \subset M$ and $\Phi : \Omega \rightrightarrows TM$ is an upper semicontinuous map such that $\Phi(x) \subset T_xM$, for all $x \in \Omega$. Since Φ is upper semicontinuous $Z(\Phi) = \{x \in \Omega : 0 \in \Phi(x)\}$ is closed in Ω. Assume that $Z(\Phi)$ is compact, therefore there exists an open relatively compact set V in S such that $Z(\Phi) \subset V \subset \mathrm{cl}V \subset \Omega$. Hence there is $\delta > 0$ such that $B(Z(\Phi), \delta) \subset V$. Therefore, along the same lines as [10, Lemma 2.4], there is $\varepsilon_0 > 0$ such that for any $x \in \mathrm{cl}V$, if $y \in B(x, \varepsilon_0)$ and $z_y \in \Phi(y)$ with $\|z_y\| < \varepsilon_0$, then $x \in B(Z(\Phi), \delta) \subset V$. In what follows $r : U \to S$ is a retraction for S with the constant L and $\varepsilon_1 := \frac{\varepsilon_0}{\max\{(C+1)(3L+4L^2)+1,(C+1)L+1\}}$.

In order to be able to define the degree of $\Phi : \Omega \rightrightarrows TM$, we need the following assumptions;

- $Z(\Phi)$ is compact.

- $\Phi : \Omega \to TM$ is upper semicontinuous with compact convex values and $\Phi(x) \cap T_S(x) \neq \emptyset$ for all $x \in \Omega$.

Note that by Theorem 2.2, it follows that the map $\psi : S \rightrightarrows TM$ defined by

$$\psi(x) := \{v \in T_x M : \ d_S^\circ(x, v) < \varepsilon\},$$

has convex values with an open graph, and hence it is lower semicontinuous. Theorem 2.4 implies that $T_S(x) \subseteq \psi(x)$, so if $\Phi(x) \cap T_S(x) \neq \emptyset$, then $\psi(x) \cap \Phi(x) \neq \emptyset$, for every $x \in S$. It follows from Theorem 3.3 that there is a vector field f such that the following hold:

(a) There exists $z_x \in \psi(x)$ such that $\|z_x - f(x)\| < \varepsilon$.

(b) There exist $\bar{x} \in B(x, \varepsilon)$ and $y_{\bar{x}} \in \Phi(\bar{x})$ such that $\|L_{\bar{x}x}(y_{\bar{x}}) - f(x)\| < \varepsilon$.

We are now ready to prove our next lemma.

LEMMA 3.5. *Suppose that S is a compact L-retract in a complete Riemannian manifold M and Ω is a nonempty open set in S. Let $\Phi : \Omega \rightrightarrows TM$ be an upper semicontinuous map with convex compact values such that $\Phi(x) \subset T_x M$, for all $x \in \Omega$. Then there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that for any $\varepsilon \in (0, \varepsilon_2]$ and any two ε-approximations $f_0,\ f_1$ of Φ and $\lambda \in [0,1]$, $f_\lambda : \mathrm{cl}V \to TM$ defined by $f_\lambda(x) = (1-\lambda) f_0(x) + \lambda f_1(x)$ is an ε_1-approximation of Φ.*

PROOF. Let $x \in \mathrm{cl}V$ be arbitrary. Since $y \mapsto L_{yx}(\Phi(y))$ is upper semicontinuous on a nieghborhood of x, there exists $\varepsilon' > 0$ such that for every $y \in B(x, \varepsilon')$, $L_{yx}(\Phi(y)) \subseteq \Phi(x) + \frac{\varepsilon_1}{4} B_{T_x M}$. Now set $\varepsilon_2 \leq \min\{\frac{\varepsilon'}{2}, \frac{\varepsilon_1}{2}\}$ and $\varepsilon \in (0, \varepsilon_2]$. Let $f_0,\ f_1$ be ε-approximations of Φ, i.e. there exist $x',\ x'' \in B(x, \varepsilon)$ and $y_{x'} \in \Phi(x'),\ y_{x''} \in \Phi(x'')$ such that

$$\|f_0(x) - L_{x'x}(y_{x'})\| < \varepsilon,\ \|f_1(x) - L_{x''x}(y_{x''})\| < \varepsilon.$$

The upper semicontinuity of $y \mapsto L_{yx}(\Phi(y))$ implies that $L_{x'x}(\Phi(x')) \subseteq \Phi(x) + \frac{\varepsilon_1}{4} B_{T_x M}$ and $L_{x''x}(\Phi(x'')) \subseteq \Phi(x) + \frac{\varepsilon_1}{4} B_{T_x M}$. Hence, there exist $z_x,\ w_x \in \Phi(x)$ such that

$$\|z_x - L_{x'x}(y_{x'})\| < \frac{\varepsilon_1}{4},\ \|w_x - L_{x''x}(y_{x''})\| < \frac{\varepsilon_1}{4}.$$

Since Φ has convex values, it follows that $(1-\lambda) w_x + \lambda z_x \in \Phi(x)$ and

$$\begin{aligned}
&\|(1-\lambda) f_0(x) + \lambda f_1(x) - ((1-\lambda) z_x + \lambda w_x)\| \leq \\
&(1-\lambda)\|f_0(x) - L_{x'x}(y_{x'})\| + \lambda \|f_1(x) - L_{x''x}(y_{x''})\| + \\
&(1-\lambda)\|z_x - L_{x'x}(y_{x'})\| + \lambda \|w_x - L_{x''x}(y_{x''})\| \leq \varepsilon + \frac{\varepsilon_1}{2} \leq \varepsilon_1,
\end{aligned}$$

as required.

THEOREM 3.6. *Let $0 < \varepsilon \leq \varepsilon_1$, and $f : \mathrm{cl}V \times [0,1] \to TM$ be a continuous function such that for any $\lambda \in [0,1]$, $f_\lambda(.) = f(., \lambda)$ is an ε- approximation of Φ. Then there is $\eta > 0$ such that for all $t \in (0, \eta)$, $x \in \mathrm{cl}V$ and $\lambda \in [0,1]$:*

(i) There is $y_x \in B(f_\lambda(x), \varepsilon)$ such that $d_S(\exp_x(t y_x)) < t\varepsilon$.

(ii) $\exp_x(tf_\lambda(x)) \in U$.

(iii) If $x = r(\exp_x(tf_\lambda(x)))$, *then* $x \in B(Z(\Phi), \delta) \subset V$, *where* $\delta > 0$ *is defined in Remark 3.4 .*

PROOF. (i) Arguing by contradiction, suppose that for any $n \in \mathbb{N}$, there exist $t_n \in (0, \frac{1}{n})$ and $x_n \in \mathrm{cl}V$ and $\lambda_n \in [0,1]$ such that if $y_{x_n} \in B(f_{\lambda_n}(x_n), \varepsilon)$, then $d_S(\exp_{x_n}(t_n y_{x_n})) \geq t_n\varepsilon$. Since cl$V$ and $[0,1]$ are compact, there are $x \in \mathrm{cl}V,\ \lambda \in [0,1]$ such that $x_n \to x$ and $\lambda_n \to \lambda$. Note that f_λ is an ε-approximation of Φ. Hence there exists $y_x \in B(f_\lambda(x), \varepsilon)$ such that $d_S^\circ(x, y_x) < \varepsilon$. We choose w_n from the definition of generalized directional derivative such that

$$d(\exp_x^{-1} \circ \exp_{x_n})(0_{x_n})(w_n) = y_x.$$

For large n, $w_n \in B(f_{\lambda_n}(x_n), \varepsilon)$, hence $t_n\varepsilon > d_S(\exp_{x_n}(t_n w_n)) \geq t_n\varepsilon$, a contradiction.

(ii) For every $x \in \mathrm{cl}V$, let $B(x, r_x)$ be a geodesic ball around x. By the compactness of cl$V$, $\mathrm{cl}V \subseteq \bigcup_{i=1}^n B(x_i, r_i)$. Now define the continuous map $g : \overline{B(x_i, r_i)} \cap \mathrm{cl}V \to T_{x_i}M$ by $g(x) = L_{xx_i}(f_\lambda(x))$. Hence there exists M_i such that $\|L_{xx_i}(f_\lambda(x))\|_{x \in \overline{B(x_i, r_i)} \cap \mathrm{cl}V} \leq M_i$. Set $M_0 := \max\{M_i :\ i = 1, ..., n\}$. For every $x \in \mathrm{cl}V$, there exists x_i such that $x \in B(x_i, r_i)$ and $\|f_\lambda(x)\| = \|L_{xx_i}(f_\lambda(x))\| \leq M_0$. Note that $t\|f_\lambda(x)\| \to 0$ as $t \to 0$. Therefore $\exp_x(tf_\lambda(x)) \to x \in U$.

(iii) If $x \in \mathrm{cl}V$ and $r(\exp_x(tf_\lambda(x))) = x$, then (i) implies that there exists $y_x \in B(f_\lambda(x), \varepsilon)$ such that

$$\begin{aligned}
&\|f_\lambda(x)\| = t^{-1}d(\exp_x(tf_\lambda(x)), x) = t^{-1}d(\exp_x(tf_\lambda(x)), r(\exp_x(tf_\lambda(x))))\\
&\leq t^{-1}Ld_S(\exp_x(tf_\lambda(x))) \leq t^{-1}Ld_S(\exp_x(ty_x)) + t^{-1}Ld(\exp_x(ty_x), \exp_x(tf_\lambda(x)))\\
&\leq t^{-1}Ld_S(\exp_x(ty_x)) + LC\|y_x - f_\lambda(x)\|\\
&\leq (C+1)L\varepsilon.
\end{aligned}$$

Note that η can be chosen as $tf_\lambda(x), ty_x \in B(0, \delta)$, where δ is such that $\exp_x$ is $C-$Lipschitz on $B(0, \delta)$. On the other hand there exist $x' \in B(x, \varepsilon)$ and $y_{x'} \in \Phi(x')$ such that $\|f_\lambda(x) - L_{x'x}(y_{x'})\| < \varepsilon$. Hence $x \in B(Z(\Phi), \delta)$.

Take $\varepsilon \in (0, \varepsilon_1)$ and let $f : \mathrm{cl}V \to TM$ be an arbitrary ε-approximation of Φ. By Theorem 3.6, for $t \in (0, \eta)$, the single valued map $g_t^\varepsilon : \mathrm{cl}V \to S$ defined by $g_t^\varepsilon(x) := r(\exp_x(tf(x))$ is well defined.

Now we define

$$\deg(\Phi, \Omega) := \lim_{\varepsilon, t \to 0^+} i(g_t^\varepsilon, V). \tag{3.1}$$

Along the same lines as [10, Lemma 2.7 and Lemma 2.8], for $0 < t_1 < t_2 < \eta$, one has $i(g_{t_1}^\varepsilon, V) = i(g_{t_2}^\varepsilon, V)$. Moreover, if f_0 and f_1 are ε-approximations of Φ, then for all $0 < t < \eta$, $i(g_0, V) = i(g_1, V)$, where $g_i : \mathrm{cl}V \to S$ are defined by $g_i(x) := r(\exp_x(tf_i(x)))$, for $i = 0, 1$. This definition does not depend on the choice of ε-approximation and stabilizes when $0 < t < \eta$. Moreover, It does not depend on the choice of r. Indeed, suppose that f is an ε-approximation of Φ, $g(x) := r(\exp_x(tf(x)))$ and $g(x)' := r'(\exp_x(tf(x)))$, where $t \in (0,\eta)$, $x \in \mathrm{cl}V$. For $x \in \mathrm{cl}V$, assume that $\gamma_x : [0,1] \to S$ is a geodesic connecting $r(\exp_x(tf(x)))$ and $r'(\exp_x(tf(x)))$. Since M is complete, for $\lambda \in [0,1]$, there exists $v \in T_{g(x)}M$ such that $\gamma_x(\lambda) := \exp_{g(x)}(\lambda v)$ connects $g(x)$ and $g'(x)$. Without loss of generality we can suppose that $\gamma_x(\lambda) \in U,\ \lambda \in [0,1]$, because

$$\begin{aligned} d_S(\gamma_x(\lambda)) &\le d_S(\exp_x(tf(x))) + d(\exp_{g(x)}(\lambda v), \exp_x(tf(x))) \\ &\le d_S(\exp_x(tf(x))) + d(\exp_{g(x)}(\lambda v), g(x)) + d(\exp_x(tf(x)), g(x)) \\ &\le d_S(\exp_x(tf(x))) + \lambda d(g(x), g'(x)) + Ld_S(\exp_x(tf(x))) \\ &\le d_S(\exp_x(tf(x))) + \lambda d(g(x), \exp_x(tf(x))) + \lambda d(g'(x), \exp_x(tf(x))) \\ &+ Ld_S(\exp_x(tf(x))) \\ &\le (1+3L)d_S(\exp_x(tf(x))) \le (1+3L)d(x, \exp_x(tf(x))) \\ &= (1+3L)t\|f(x)\| \to 0, \quad \text{as} \quad t \to 0. \end{aligned}$$

Now let $h(x,\lambda) := r(\gamma_x(\lambda))$, $x \in \mathrm{cl}V,\ \lambda \in [0,1]$. If $x = h(x,\lambda)$, for $x \in \mathrm{cl}V,\ \lambda \in [0,1]$, then

$$\begin{aligned} &\|f(x)\| = t^{-1}d(h(x,\lambda), \exp_x(tf(x))) \\ &\le t^{-1}d(\exp_x(tf(x)), \gamma_x(\lambda)) + t^{-1}d(h(x,\lambda), \gamma_x(\lambda)) \\ &\le t^{-1}d(\exp_x(tf(x)), r(\exp_x(tf(x)))) + t^{-1}d(r(\exp_x(tf(x))), \gamma_x(\lambda)) \\ &+ t^{-1}d(h(x,\lambda), \gamma_x(\lambda)) \\ &\le t^{-1}d(\exp_x(tf(x)), r(\exp_x(tf(x)))) + \lambda t^{-1}d(g'(x), g(x)) + t^{-1}d(h(x,\lambda), \gamma_x(\lambda)) \\ &\le t^{-1}d(\exp_x(tf(x)), r(\exp_x(tf(x)))) + t^{-1}\lambda d(\exp_x(tf(x)), g(x)) \\ &+ t^{-1}\lambda d(\exp_x(tf(x)), g'(x)) + t^{-1}d(h(x,\lambda), \gamma_x(\lambda)) \\ &\le t^{-1}3Ld_S(\exp_x(tf(x))) + t^{-1}Ld_S(\gamma_x(\lambda)) \le t^{-1}(4L+3L^2)d_S(\exp_x(tf(x))). \end{aligned}$$

By Theorem 3.6, there is $y_x \in B(f(x),\varepsilon)$ such that

$$\begin{aligned} &t^{-1}(3L^2+4L)d_S(\exp_x(tf(x))) \\ &\qquad \le t^{-1}(3L^2+4L)(d_S(\exp_x(ty_x) + d(\exp_x(tf(x)), \exp_x(ty_x)))) \\ &\qquad \le t^{-1}(3L^2+4L)d_S(\exp_x(ty_x)) + (3L^2+4L)C\|f(x) - y_x\| \\ &\qquad \le (3L^2+4L)(C+1)\varepsilon. \end{aligned}$$

Thus one can deduce that $x \in B(Z(\Phi),\delta)$ and h provides a homotopy between g and g'.

Theorem 3.7. *The degree defined by (3.1) has the following properties;*

- *(Existence) If* $\deg(\Phi,\Omega) \neq 0$*, then* $Z(\Phi) \neq \emptyset$*.*

- *(additivity) If* $\Omega_1, \Omega_2 \subset \Omega$ *are open in* S *and* $Z(\Phi) \subset (\Omega_1 \cup \Omega_2) \setminus (\Omega_1 \cap \Omega_2)$, *then*
$$\deg(\Phi, \Omega) = \deg(\Phi|_{\Omega_1}, \Omega_1) + \deg(\Phi|_{\Omega_1}, \Omega_1).$$
- *(Normalization)* $\deg(\Phi, S) = \chi(S)$.
- *(Homotopy invariance) Assume that* $\Phi_0, \Phi_1 : \Omega \rightrightarrows TM$ *are homotopic in the sense that there is an upper semicontinuous map* $\Phi : \Omega \rightrightarrows TM$ *with compact convex values such that* $\Phi(., i) = \Phi_i$, $i = 0, 1$, *for all* $x \in \Omega$ *and* $\lambda \in [0, 1]$, $\Phi(x, \lambda) \in T_xM$ *and* $\Phi(x, \lambda) \cap T_S(x) \neq \emptyset$ *and* $\{x \in \Omega |\ x \in \Phi(x, \lambda)$ *for some* $\lambda \in [0, 1]\}$ *is compact.*

PROOF. We only need to prove the first property, because other statements can be proved along the same lines as Proposition 2.10 in [10]. To prove the first property; assume that $\deg(\Phi, \Omega) \neq 0$, then for t, ε small enough, $i(g_t^\varepsilon, V) \neq 0$. Therefore, there is $x \in \mathrm{cl}V$ such that $x = r(\exp_x(tf(x)))$. Assume that $y_x \in B(f(x), \varepsilon)$ such that $d_S(\exp_x(ty_x)) < t\varepsilon$, then

$$\begin{aligned}
&\|f(x)\| = t^{-1}d(\exp_x(tf(x)), x) = t^{-1}d(\exp_x(tf_\lambda(x)), r(\exp_x(tf(x))))\\
&\leq t^{-1}Ld_S(\exp_x(tf(x))) \leq t^{-1}Ld_S(\exp_x(ty_x)) + t^{-1}Ld(\exp_x(ty_x), \exp_x(tf(x)))\\
&\leq t^{-1}Ld_S(\exp_x(ty_x)) + LC\|y_x - f(x)\|\\
&\leq (C + 1)L\varepsilon.
\end{aligned}$$

Using the compactness of $\mathrm{cl}V$ and the upper semicontinuity of Φ, we have $Z(\Phi) \neq \emptyset$.

It is now obvious by the existence and normalization properties of the degree that if S is a compact L-retract and $\chi(S)$ is nontrivial, then Φ has an equilibrium.

THEOREM 3.8. *Let* S *be a compact* L*-retract subset of a complete Riemannian manifold* M *with nontrivial Euler characteristic. Suppose that* $\Phi : S \rightrightarrows TM$ *is an upper semicontinuous map with compact convex values such that*

$$\Phi(x) \subset T_xM,\ \Phi(x) \cap T_S(x) \neq \emptyset \quad \textit{for all } x \in S.$$

Then Φ *has an equilibrium.*

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