PSEUDO-JACOBIAN AND CHARACTERIZATION OF MONOTONE VECTOR FIELDS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, a notion of pseudo-Jacobian of continuous vector fields on Riemannian manifolds is presented. It is shown that the Clarke generalized Jacobian and Mordukhovich coderivative for locally Lipschitz vector fields are pseudo-Jacobians. Moreover, monotone vector fields are characterized in terms of pseudo-Jacobians.

1. Introduction

As many problems in computer vision, robotics, signal processing and geometric mechanics are expressed as nonsmooth problems on Riemannian manifolds, the huge impact of developing nonsmooth analysis concepts on manifold settings to solve these problems is undeniable. Therefore it is of eminent interest to develop useful computational and theoretical tools for nonsmooth objects such as nondifferentiable functions, vector fields and forms on manifolds. In the past few years a number of results have been obtained on numerous aspects of nonsmooth analysis on Riemannian manifolds; see, e.g., [1, 2, 6, 7, 8, 13, 14].

Research in the area of nonsmooth analysis of vector-valued maps and their generalized Jacobian matrices such as the Clarke generalized Jacobian matrices of locally Lipschitz maps and Mordukhovich coderivative for general vector-valued maps on Banach spaces has been of substantial interest in recent years; see for instance [4, 15]. Hence the development and analysis of concepts for nonsmooth vector fields, which can be viewed as a natural generalization of nonsmooth vector-valued maps, are essential from both theoretical and numerical viewpoints.

The concept of pseudo-Jacobian (also called approximate Jacobian) matrices, which is a generalization of the idea of convexificators of real-valued

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functions, extends the nonsmooth analysis of locally Lipschitz maps to continuous maps and unifies various results of nonsmooth analysis; see [9, 10]. The monotonicity of vector-valued maps plays a crucial role in the study of complementarity problems, variational inequality problems, and equilibrium problems. In [11] characterizations of generalized monotonicity in terms of pseudo-Jacobian matrices are presented. On the other hand in recent years a great deal of research has focused on the study of the notion of monotonicity for vector fields on manifolds and a number of important results have been obtained on various aspects of optimization theory and applications for problems formulated on Riemannian manifolds; see [5, 14, 16] and the references cited therein.

The aim of this paper is to present a notion of pseudo-Jacobian for continuous vector fields on manifolds, and to use this notion to characterize monotone vector fields on manifolds. Moreover, we show that pseudo-Jacobian characterizes the first and second order optimality conditions in easily verifiable forms. The rest of the paper is organized as follows. Section 2 is devoted to the definition of pseudo-Jacobian associated to a continuous vector field on a Riemannian manifold and some basic properties are presented. In Section 3 characterizations of monotonicity and generalized monotonicity of continuous vector fields via pseudo-Jacobian are presented.

2. Pseudo-Jacobian of continuous vector fields

Let us first introduce some standard notations and known results of Riemannian manifolds, see, e.g. [12, 18]. Throughout this paper, M is an n-dimensional complete, connected manifold endowed with a Riemannian metric $\langle .,. \rangle_x$ on the tangent space T_xM. The corresponding norm is denoted by $\| \|_x$. As usual we denote by $B(x, \delta)$ the open ball centered at x with radius δ, by int N (cl N) the interior (closure) of the set N and by co N the convex hull of N.

Recall that the set S in a Riemannian manifold M is called convex if every two points $p_1,\ p_2 \in S$ can be joined by a unique geodesic whose image belongs to S. We identify (via the Riemannian metric) the tangent space of M at a point x, denoted by T_xM, with the cotangent space at x, denoted by T_x^*M. For the point $x \in M$, $\exp_x : U_x \to M$ will stand for the exponential function at x, where U_x is an open subset of T_xM. Recall that $\exp_x$ maps straight lines of the tangent space T_xM passing through $0_x \in T_xM$ into geodesics of M passing through x.

We will also use the parallel transport of vectors along geodesics. Recall that, for a given curve $\gamma : I \to M$, number $t_0 \in I$, and a vector $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = V_0$. Moreover, the map defined by $V_0 \mapsto V(t_1)$ is a linear isometry between the tangent spaces $T_{\gamma(t_0)}M$ and $T_{\gamma(t_1)}M$, for each $t_1 \in I$. In the case when γ is a minimizing geodesic and $\gamma(t_0) = x, \gamma(t_1) = y$, we will denote this map by $L_{xy,\gamma}$, and we will call it the parallel transport from T_xM to T_yM along the curve γ. In what follows, L_{xy} will be used when the minimizing geodesic which connects x to y, is unique.

Recall that the Hessian $D^2\varphi$ of a C^2 smooth function φ on a Riemannian manifold M is defined by

$$D^2\varphi(X,Y) = \langle \nabla_X \nabla\varphi, Y\rangle,$$

where $\nabla\varphi$ is the gradient of φ, X, Y are vector fields on M and $\nabla_Y X$ denotes the covariant derivative of X along Y(see [18, p. 31]). The Hessian is a symmetric tensor field of type $(0,2)$ and, for a point $p \in M$, the value $D^2\varphi(X,Y)(p)$ only depends on φ and the vectors $X(p)$, $Y(p) \in T_pM$. So we can define the second derivative of φ at p as the symmetric bilinear form $d^2\varphi(p) : T_pM \times T_pM \to \mathbb{R}$,

$$(v,w) \mapsto d^2\varphi(p)(v,w) := D^2\varphi(X,Y)(p),$$

where X, Y are any vector fields such that $X(p) = v$, $Y(p) = w$. A useful way to compute $d^2\varphi(p)(v,v)$ is to take geodesic γ whit $\gamma'(0) = v$ and calculate

$$\frac{d^2}{dt^2}\varphi(\gamma(t))\Big|_{t=0}.$$

We will often write $d^2\varphi(p)(v)^2$ instead of $d^2\varphi(p)(v,v)$.

To define the notion of pseudo-Jacobian associated to a continuous vector field on a Riemannian manifold, we confront with several technical issues occurred by working on double tangent bundles. We first recall some basic properties of double tangent bundle TTM. Let $\pi : TM \to M$ denote the canonical projection, then the differential of π denoted by $d\pi$ is a map from TTM onto TM. If $X(x) \in TM$, then we denote the kernel of $d\pi$ at $X(x)$ by $\mathcal{V}_{X(x)}$ and call it the vertical subspace of $T_{X(x)}TM$. This vertical subspace is of dimension n, therefore we identify the vertical subspace of $T_{X(x)}TM$ with T_xM. Note that $\mathcal{L}(T_xM, \mathcal{V}_{X(x)})$ denotes the n^2-dimensional affine subspace of the space of all linear mappings from T_xM to $T_{X(x)}TM$ denoted by $\mathcal{L}(T_xM, T_{X(x)}TM)$.

Now we are ready to define the notion of pseudo-Jacobian for a continuous vector field. This notion was introduced in [9] for continuous vector-valued functions on linear spaces, where it was called approximate Jacobian.

Assume that $X : M \to TM$ is a continuous vector field on a Riemannian manifold M. For each $x \in M$ and $v \in T_xM$, we define a function $vX : B(x,r) \to \mathbb{R}$ by

$$(vX)(y) := \langle v, L_{yx}(X(y))\rangle_x, \tag{2.1}$$

where $B(x,r)$ is a geodesic ball around x.

The lower Dini directional derivative and upper Dini directional derivative of vX at x in the direction $u \in T_xM$ are defined by

$$(vX)^-(x,u) := \liminf_{t\downarrow 0} \frac{\langle v, L_{\gamma(t)x}(X(\gamma(t))) - X(x)\rangle_x}{t},$$

$$(vX)^+(x,u) := \limsup_{t\downarrow 0} \frac{\langle v, L_{\gamma(t)x}(X(\gamma(t))) - X(x)\rangle_x}{t},$$

where γ is a geodesic starting at x with $\gamma'(0) = u$.

Definition 2.1. (Pseudo-Jacobian) *Assume that $X : M \to TM$ is a continuous vector field on a Riemannian manifold M. A nonempty closed subset of $\mathcal{L}(T_xM, \mathcal{V}_{X(x)})$ is called a pseudo-Jacobian of X at x, denoted by $\partial^* X(x)$, if for every $v \in T_xM$ one has*

$$(vX)^+(x,u) \leq \sup_{A\in\partial^* X(x)} \langle v, Au\rangle_x \qquad \forall u \in T_xM. \tag{2.2}$$

If equality holds in (2.2), we say that $\partial^* X(x)$ is a regular pseudo-Jacobian of X at x

Note that condition (2.2) is equivalent to the condition that

$$(vX)^-(x,u) \geq \inf_{A\in\partial^* X(x)} \langle v, Au\rangle_x \qquad \forall u \in T_xM. \tag{2.3}$$

It is worthwhile to mention that if $\partial^* X(x) \subset \mathcal{L}(T_xM, \mathcal{V}_{X(x)})$ is a pseudo-Jacobian of X at x, then every closed subset $B \subseteq \mathcal{L}(T_xM, \mathcal{V}_{X(x)})$ containing $\partial^* X(x)$ is a pseudo-Jacobian of X at x.

Let us now introduce the notion of pseudo-Hessian for functions defined on Riemannian manifolds.

Definition 2.2. (Pseudo-Hessian) *Let $f : M \to \mathbb{R}$ be a continuously differentiable function on a Riemannian manifold M. Then the gradient vector field $\nabla f : M \to TM$ is continuous. We say that a closed subset $\partial^2_* f(x)$ of $\mathcal{L}(T_xM, \mathcal{V}_{\nabla f(x)})$ is a pseudo-Hessian of f at x if it is a pseudo-Jacobian of ∇f at x.*

Example 2.3. *If $X : M \to TM$ is a smooth vector field at $x \in M$, then any closed subset of $\mathcal{L}(T_xM, \mathcal{V}_{X(x)})$ containing the linear operator $DX(x) : T_xM \to T_xM$ defined by $DX(x)u = (\nabla_u X)(x)$, is a pseudo-Jacobian of X at x. Since for each $v,\ u \in T_xM$,*

$$\begin{aligned}(vX)^+(x,u) &= \limsup_{t\downarrow 0} \frac{\langle v, L_{\gamma(t)x}(X(\gamma(t))) - X(x)\rangle_x}{t} \\ &= \langle v, D_t(X\circ\gamma)(t)|_{t=0}\rangle_x = \langle v, (\nabla_{\gamma'(0)} X)(x)\rangle_x \\ &= \langle v, (\nabla_u X)(x)\rangle_x = \langle v, DX(x)u\rangle_x \leq \sup_{A\in\partial^* X(x)} \langle v, Au\rangle_x,\end{aligned}$$

where γ is a geodesic starting at x with $\gamma'(0) = u$ and D_t denotes the covariant derivative of a vector field along a curve.

Let us present more examples of pseudo-Jacobian on Riemannian manifolds. We first show that the Clarke generalized Jacobian of a locally Lipschitz vector field is a pseudo-Jacobian. Recall that $X : M \to TM$ is said to be a Lipschitz vector field of rank l near a given point $x \in M$, if for some $\varepsilon > 0$, we have

$$\|L_{yz}(X(y)) - X(z)\|_z \leq l \ \mathrm{dist}(y, z) \quad \text{for all } z, y \in B(x, \varepsilon),$$

where $B(x,\varepsilon)$ is a convex ball. Note that such a convex ball exists due to the Whitehead theorem.

Example 2.4. Suppose that $X : M \to TM$ is a locally Lipschitz vector field on M. Then the Clarke generalized Jacobian of X at x denoted by $\partial_c X(x)$ is defined by

$$\partial_c X(x) := \mathrm{co}\{\lim_{i\to\infty} DX(x_i) : \ x_i \to x, \ x_i \in DIFF(X)\},$$

where $DIFF(X)$ is the set of all points $x \in M$ such that X is differentiable at x. In [17] it was proved that $\partial_c X(x)$ is a subset of $\mathcal{L}(T_xM, \mathcal{V}_{X(x)})$. The Clarke generalized Jacobian $\partial_c X(x)$ is a pseudo-Jacobian of X at x. To see this, we claim that for each $v \in T_xM$, $\partial_c(vX)(x)(.) = \langle v, \partial_c X(x)(.)\rangle_x$. Since for every $x \in M$ and $\xi \in \partial_c(vX)(x)(.)$, we have $\xi = \lim_{i\to\infty} \xi_i$ such that $\xi_i = D(vX)(x_i)$, $x_i \to x$. Therefore for every smooth vector field $u : M \to TM$, $\langle \xi_i, u(x_i)\rangle_{x_i} \to \langle \xi, u(x)\rangle_x$. Let γ be a geodesic with $\gamma(t_i) = x_i$ and $\gamma'(t_i) = u(x_i)$, then

$$\begin{aligned}
\langle \xi_i, u(x_i)\rangle_{x_i} &= D(vX)(x_i)(u(x_i)) = D(vX)(\gamma(t_i))(\gamma'(t_i)) \\
&= D(vX \circ \gamma)(t)\big|_{t=t_i} = \frac{d}{dt}\big|_{t=t_i} \langle L_{x\gamma(t)}(v), X(\gamma(t))\rangle_{\gamma(t)} \\
&= \langle D_t L_{x\gamma(t)}(v), X(\gamma(t))\rangle_{\gamma(t)}\big|_{t=t_i} + \langle L_{x\gamma(t)}(v), D_t X(\gamma(t))\rangle_{\gamma(t)}\big|_{t=t_i} \\
&= \langle D_t L_{x\gamma(t_i)}(v), X(\gamma(t_i))\rangle_{\gamma(t_i)} + \langle L_{x\gamma(t_i)}(v), D_t X(\gamma(t_i))\rangle_{\gamma(t_i)}.
\end{aligned}$$

According to the properties of the parallel vector fields, we have that $D_t L_{x\gamma(t_i)}(v) = 0$. Therefore $\langle \xi_i, u(x_i)\rangle_{x_i} = \langle L_{x\gamma(t_i)}(v), D_t X(\gamma(t_i))\rangle_{x_i}$. Since $x_i \in DIFF(X)$, we have $DX(\gamma(t_i))(\gamma'(t_i)) = D_t X(\gamma(t_i))$. Hence

$$\langle \xi_i, u(x_i)\rangle_{x_i} = \langle L_{xx_i}(v), DX(x_i)(u(x_i))\rangle_{x_i} \to \langle \xi, u(x)\rangle_x,$$

which implies $\xi \in \langle v, \partial_c X(x)(.)\rangle_x$. The converse inclusion can be proved by using the fact that for every $\xi \in \langle v, \partial_c X(x)(.)\rangle_x$ and smooth vector field u, we have

$$\begin{aligned}
\langle \xi, u(x)\rangle_x &= \langle v, \lim_{i\to\infty} DX(x_i)(u(x_i))\rangle_x \\
&= \lim_{i\to\infty} \langle L_{xx_i}(v), DX(x_i)(u(x_i))\rangle_{x_i}.
\end{aligned}$$

Therefore $\xi \in \partial_c(vX)(x)(.)$. Consequently, for each $u \in T_xM$,

$$(vX)^\circ(x, u) = \max_{\xi \in \partial_c(vX)(x)} \langle \xi, u\rangle_x = \max_{A \in \partial_c X(x)} \langle v, Au\rangle_x,$$

where $(vX)^\circ(x, u)$ denotes the Clarke directional derivative of the real-valued function vX at x in the direction $u \in T_xM$; see [6]. It is easy to prove that for each $u, v \in T_xM$,

$$(vX)^+(x,u) \leq (vX)^\circ(x,u),$$

therefore the set $\partial_c X(x)$ is a pseudo-Jacobian of X at x.

Another example of a pseudo-Jacobian is the Mordukhovich coderivative of a locally Lipschitz vector field on a Riemannian manifold as defined in (2.6). To show this we need to present some concepts and tools of nonsmooth analysis on Riemannian manifolds. Let us first define the Bouligand tangent cone, Bouligand and Mordukhovich normal cones in the setting of Riemannian manifolds; see [8, 19]. Let M be a Riemannian manifold and S be a closed subset of M, and let $x \in S$ and (φ, U) be a chart of M at x. The Bouligand (or contingent) tangent cone to S at x, denoted by $T_x^B S$, is defined as follows

$$T_x^B S := d\varphi(x)^{-1}[T^B_{\varphi(x)}\varphi(S \cap U)],$$

where $T^B_{\varphi(x)}\varphi(S \cap U)$ is the Bouligand tangent cone to $\varphi(S \cap U)$ as a subset of $\mathbb{R}^n$ at $\varphi(x)$. The definition of $T_x^B S$ does not depend on the choice of the chart φ at x. Hence for any normal neighborhood U of x, we have

$$\begin{aligned} T_x^B S &:= T^B_{0_x}\exp_x^{-1}(S \cap U) \\ &= \{v \in T_xM : \ v = \lim_{j\to\infty} \frac{\exp_x^{-1} x_j}{t_j}, \ t_j \downarrow 0^+ \text{ and } x_j \xrightarrow{S \cap U} x\}, \end{aligned}$$

where $x_j \xrightarrow{S \cap U} x$ means that $x_j \to x$ and $x_j \in S \cap U$.

The Bouligand normal cone to S at x, denoted by $N_x^B S$, is defined as follows

$$N_x^B S := \{\xi \in T_xM : \ \limsup_{x_j \xrightarrow{S} x} \frac{\langle \xi, \exp_x^{-1} x_j\rangle_x}{\|\exp_x^{-1} x_j\|_x} \leq 0\}.$$

The Mordukhovich normal cone to S at x, denoted by $N_x^{\mathcal{M}} S$, is defined as follows

$$N_x^{\mathcal{M}} S := \text{Limsup}_{x_j \xrightarrow{S} x} N^B_{x_j} S.$$

Let $f : M \to \mathbb{R}$ be a lower semicontinuous function on a Riemannian manifold M and $x \in M$ with $| f(x) | < \infty$. The Mordukhovich subdifferential of f at x is defined by

$$\partial_{\mathcal{M}} f(x) := \text{Limsup}_{x_j \xrightarrow{f} x} \hat{\partial}_{\mathcal{M}} f(x_j), \tag{2.4}$$

where $x_j \xrightarrow{f} x$ means $x_j \to x, \ f(x_j) \to f(x)$ and

$$\hat{\partial}_{\mathcal{M}} f(x) := \{\xi \in T_xM : \ \liminf_{x_j \to x} \frac{f(x_j) - f(x) - \langle \xi, \exp_x^{-1} x_j\rangle_x}{\|\exp_x^{-1} x_j\|_x} \geq 0\}. \tag{2.5}$$

Suppose that $X : M \to TM$ is a continuous vector field on a Riemannian manifold M. Then the Mordukhovich coderivative of X at $x \in M$ is the set valued map $D^*_{\mathcal{M}}X(x) \subseteq \mathcal{L}(T_xM, T_xM)$ defined by

$$D^*_{\mathcal{M}}X(x)(v) := \{\xi \in T_xM : (\xi, -v) \in N^{\mathcal{M}}_{X(x)}(\text{graph}(X))\}, \tag{2.6}$$

where $v \in T_xM$ and $\text{graph}(X) := \{(x, X(x)) : x \in M\} \subseteq TM$. Note that the identification $T_xM \cong \mathcal{V}_{X(x)}$ is used.

Example 2.5. We show that the Mordukhovich coderivative of a locally Lipschitz vector field on a Riemannian manifold is a pseudo-Jacobian. First we claim that for a continuous vector field X around $x \in M$ and by identifying $\mathcal{V}_{X(x)}$ and T_xM, we have

$$\partial_{\mathcal{M}}(vX)(x) \subset D^*_{\mathcal{M}}X(x)(v) \text{ for every } v \in T_xM.$$

If in addition X is locally Lipschitz, then we have

$$\partial_{\mathcal{M}}(vX)(x) = D^*_{\mathcal{M}}X(x)(v) \text{ for every } v \in T_xM. \tag{2.7}$$

To prove the claim, let $\xi \in \partial_{\mathcal{M}}(vX)(x)$. Using (2.4), we can find sequences $x_k \to x$, $\xi_k \to \xi$ provided that $\langle \xi_k, u(x_k)\rangle_{x_k} \to \langle \xi, u(x)\rangle_x$ for every smooth vector field u and $\xi_k \in \hat{\partial}_{\mathcal{M}}(vX)(x_k)$ for $k \in \mathbb{N}$. Due to (2.5) for each k, we have

$$\liminf_{x_{k_j} \to x_k} \frac{(vX)(x_{k_j}) - (vX)(x_k) - \langle \xi_k, \exp^{-1}_{x_k} x_{k_j}\rangle_{x_k}}{\|\exp^{-1}_{x_k} x_{k_j}\|_{x_k}} \geq 0.$$

The latter implies that

$$\limsup_{x_{k_j} \to x_k} \frac{(vX)(x_k) - (vX)(x_{k_j}) + \langle \xi_k, \exp^{-1}_{x_k} x_{k_j}\rangle_{x_k}}{\|\exp^{-1}_{x_k} x_{k_j}\|_{x_k}} = \tag{2.8}$$

$$\limsup_{x_{k_j} \to x_k} \frac{\langle v, L_{x_k x}X(x_k)\rangle_x - \langle v, L_{x_{k_j} x}X(x_{k_j})\rangle_x + \langle \xi_k, \exp^{-1}_{x_k} x_{k_j}\rangle_{x_k}}{\|\exp^{-1}_{x_k} x_{k_j}\|_{x_k}} \leq 0. \tag{2.9}$$

It is worth mentioning that vX is defined on a geodesic ball $B(x, r)$, therefore x_k and x_{k_j} are also chosen in this geodesic ball. This ensures that

$$\langle v, L_{x_k x}X(x_k)\rangle_x - \langle v, L_{x_{k_j} x}X(x_{k_j})\rangle_x = \langle L_{x x_k} v, X(x_k) - L_{x_{k_j} x_k}X(x_{k_j})\rangle_{x_k}.$$

Therefore from (2.8) one can deduce that

$$\limsup_{x_{k_j} \to x_k} \frac{\langle \xi_k, \exp^{-1}_{x_k} x_{k_j}\rangle_{x_k} - \langle L_{x x_k} v, L_{x_{k_j} x_k}X(x_{k_j}) - X(x_k)\rangle_{x_k}}{\|\exp^{-1}_{x_k} x_{k_j}\|_{x_k}} \leq 0. \tag{2.10}$$

Using (2.10), we prove that $(\xi_k, -L_{x x_k} v) \in N^B_{X(x_k)}\text{graph}(X)$ for each $k \in \mathbb{N}$, which implies that $\xi \in D^*_{\mathcal{M}}X(x)(v)$ due to the coderivative definition in (2.6). To prove that $(\xi_k, -L_{x x_k} v) \in N^B_{X(x_k)}\text{graph}(X)$, assume that Φ is a chart of TM at $X(x_k)$ defined by $\Phi : U(X(x_k)) \subset TM \to W \subset T_{x_k}M \times T_{x_k}M$, $\Phi(y, v_y) = (\exp^{-1}_{x_k}(y), d\exp^{-1}_{x_k}(y)(v_y))$. Then identifying $T_{X(x_k)}(TM)$ and $T_{x_k}M \oplus T_{x_k}M$, we have

$$N^B_{X(x_k)}\text{graph}(X) = N^B_{(0_{x_k}, X(x_k))}\Phi(\text{graph}(X) \cap U(X(x_k))),$$

and $N^B_{(0_{x_k},X(x_k))}\Phi(\mathrm{graph}(X)\cap U(X(x_k)))$ contains all $(\xi_1,\xi_2)\in T_{x_k}M\times T_{x_k}M$ satisfying the following condition;

$$\limsup_{x_{k_j}\to x_k}\frac{\langle \xi_1,\exp^{-1}_{x_k}(x_{k_j})\rangle_{x_k}+\langle \xi_2, d\exp^{-1}_{x_k}(x_{k_j})(X(x_{k_j}))-X(x_k)\rangle_{x_k}}{\|(\exp^{-1}_{x_k}(x_{k_j}), d\exp^{-1}_{x_k}(x_{k_j})(X(x_{k_j}))-X(x_k))\|_{(x_k,x_k)}}\leq 0.$$

Now, considering [3, Proposition 4.1], which states that $d\exp^{-1}_{x_k}(x_{k_j})\to L_{x_{k_j}x_k}$ whenever $x_{k_j}\to x_k$, and using (2.10), we conclude that $(\xi_k,-L_{xx_k}v)\in N^B_{X(x_k)}\mathrm{graph}(X)$ for each $k\in\mathbb{N}$.

Conversely, we pick $\xi\in D^*_{\mathcal{M}}X(x)(v)$, therefore there are sequences $v_k\to v$, $x_k\to x$ and $\xi_k\to\xi$ such that $(\xi_k,-v_k)\in N^B_{X(x_k)}(\mathrm{graph}(X))$ for $k\in\mathbb{N}$. Hence

$$\limsup_{x_{k_j}\to x_k}\frac{\langle \xi_k,\exp^{-1}_{x_k}(x_{k_j})\rangle_{x_k}-\langle v_k, L_{x_{k_j}x_k}X(x_{k_j})-X(x_k)\rangle_{x_k}}{\|(\exp^{-1}_{x_k}(x_{k_j}), L_{x_{k_j}x_k}(X(x_{k_j}))-X(x_k))\|_{(x_k,x_k)}}\leq 0,$$

therefore there exists η_k such that for all $x_{k_j}\in B(x_k,\eta_k)$

$$\langle \xi_k,\exp^{-1}_{x_k}x_{k_j}\rangle_{x_k}-\langle v_k, L_{x_{k_j}x_k}X(x_{k_j})-X(x_k)\rangle_{x_k}<\frac{1}{k}(l+1)\|\exp^{-1}_{x_k}x_{k_j}\|_{x_k},$$

where $l>0$ is a Lipschitz constant of X around x. Therefore

$$\begin{aligned}\langle \xi_k,\exp^{-1}_{x_k}x_{k_j}\rangle_{x_k}&-\langle L_{xx_k}v, L_{x_{k_j}x_k}X(x_{k_j})-X(x_k)\rangle_{x_k}\\&+\langle L_{xx_k}v-v_k, L_{x_{k_j}x_k}X(x_{k_j})-X(x_k)\rangle_{x_k}\\&<\frac{1}{k}(l+1)\|\exp^{-1}_{x_k}x_{k_j}\|_{x_k}.\end{aligned}$$

Then by the definition of locally Lipschitz vector fields, we get

$$\begin{aligned}\langle \xi_k,\exp^{-1}_{x_k}x_{k_j}\rangle_{x_k}&-\langle L_{xx_k}v, L_{x_{k_j}x_k}X(x_{k_j})-X(x_k)\rangle_{x_k}\\&<\|\exp^{-1}_{x_k}x_{k_j}\|_{x_k}(\frac{1}{k}+\frac{l}{k}+\|v_k-L_{xx_k}v\|_{x_k}l).\end{aligned}$$

This results that $\xi_k\in\hat{\partial}_{\mathcal{M}}(vX)(x_k)$, which yields $\xi\in\partial_{\mathcal{M}}(vX)(x)$ due to (2.4) and the proof of the claim is completed.

According to the definition of the Clarke normal cone in [19], we have that $N^C_xS=\mathrm{co}N^{\mathcal{M}}_xS$ and hence if $f:M\to\mathbb{R}$ is Lipschitz continuous around x, then $\partial_Cf(x)=\mathrm{co}\partial_{\mathcal{M}}f(x)$. Now concluding from (2.7) and discussion in Example 2.4, the Mordukhovich coderivative satisfies the following equality

$$\langle v,\partial_cX(x)(.)\rangle_x=\mathrm{co}(\langle D^*_{\mathcal{M}}X(x)(v),.\rangle_x)\text{ for all }v\in T_xM. \tag{2.11}$$

Finally by applying (2.11) and the fact that $\partial_cX(x)$ is a pseudo-Jacobian of X at x, we can say that the Mordukhovich coderivative $D^*_{\mathcal{M}}X(x)$ is a pseudo-Jacobian of X at this point.

We now proceed to provide elementary rules for pseudo-Jacobian.

Theorem 2.6. (Scalar multiples and sums) *Let X and Y be two continuous vector fields. If $\partial^* X(x)$ and $\partial^* Y(x)$ are pseudo-Jacobian of X and Y, respectively, at x, then*

- *$\alpha\partial^* X(x)$ is a pseudo-Jacobian of αX at x for every $\alpha \in \mathbb{R}$.*
- *$\mathrm{cl}(\partial^* X(x) + \partial^* Y(x))$ is a pseudo-Jacobian of $X + Y$ at x.*

Proof. Let $\alpha \in \mathbb{R}$. If $\alpha \geq 0$, then for every $u, v \in T_xM$, we have

$$\begin{aligned}(v(\alpha X))^+(x,u) = \alpha(vX)^+(x,u) &\leq \alpha \sup_{A\in\partial^* X(x)} \langle v, Au\rangle_x \\ &\leq \sup_{A\in\partial^* X(x)} \langle v, \alpha Au\rangle_x \leq \sup_{\tilde{A}\in\alpha\partial^* X(x)} \langle v, \tilde{A}u\rangle_x,\end{aligned}$$

which shows that $\alpha\partial^* X(x)$ is a pseudo-Jacobian of αX at x.

If $\alpha < 0$, we have

$$\begin{aligned}(v(\alpha X))^+(x,u) = -\alpha(-vX)^+(x,u) &\leq -\alpha \sup_{A\in\partial^* X(x)} \langle -v, Au\rangle_x \\ &\leq \sup_{A\in\partial^* X(x)} \langle v, \alpha Au\rangle_x \leq \sup_{\tilde{A}\in\alpha\partial^* X(x)} \langle v, \tilde{A}u\rangle_x,\end{aligned}$$

as required.

To prove the second property, assume that $u, v \in T_xM$,

$$\begin{aligned}(v(X+Y))^+(x,u) &\leq (vX)^+(x,u) + (vY)^+(x,u) \\ &\leq \sup_{A\in\partial^* X(x)} \langle v, Au\rangle_x + \sup_{\tilde{A}\in\partial^* Y(x)} \langle v, \tilde{A}u\rangle_x \\ &\leq \sup_{P\in\partial^* X(x)+\partial^* Y(x)} \langle v, Pu\rangle_x,\end{aligned}$$

which shows that the closure of the set $\partial^* X(x)+\partial^* Y(x)$ is a pseudo-Jacobian of $X + Y$ at x. □

3. Monotone vector fields on Riemannian manifolds

A valuable concept in the study of mappings that appear in many problems, such as optimization, equilibrium or in variational inequality problems is monotonicity. In this section we characterize the monotonicity and generalized monotonicity of continuous vector fields using pseudo-Jacobian. Moreover, we present optimality conditions in terms of pseudo-Jacobian.

Definition 3.1. *Let M be a Riemannian manifold and X be a vector field on M.*

(a) The vector field X is said to be monotone if and only if for every x, $y \in M$,

$$\langle \gamma'(0), L_{yx,\gamma}(X(y)) - X(x)\rangle_x \geq 0, \tag{3.1}$$

where γ is a geodesic joining x and y.

(b) The vector field X is said to be strictly monotone if and only if for every $x,\ y \in M$,

$$\langle \gamma'(0), L_{yx,\gamma}(X(y)) - X(x)\rangle_x > 0, \tag{3.2}$$

where γ is a geodesic joining x and y.

(c) The vector field X is said to be trivially monotone if and only if for every $x,\ y \in M$,

$$\langle \gamma'(0), L_{yx,\gamma}(X(y)) - X(x)\rangle_x = 0, \tag{3.3}$$

where γ is a geodesic joining x and y.

(d) The vector field X is said to be quasimonotone if for each $x,\ y \in M$,

$$\langle X(x), \gamma'(0)\rangle_x > 0 \qquad \textit{implies} \qquad \langle X(y), \gamma'(1)\rangle_y \geq 0. \tag{3.4}$$

where γ is a geodesic joining x and y.

(e) The vector field X is said to be pseudomonotone if for each $x,\ y \in M$,

$$\langle X(x), \gamma'(0)\rangle_x > 0 \qquad \textit{implies} \qquad \langle X(y), \gamma'(1)\rangle_y > 0. \tag{3.5}$$

where γ is a geodesic joining x and y.

In view of (3.5), the strict inequalities can be replaced by inequalities

$$\langle X(x), \gamma'(0)\rangle_x \geq 0 \qquad \text{implies} \qquad \langle X(y), \gamma'(1)\rangle_y \geq 0. \tag{3.6}$$

To characterize the monotoncity of a vector field, a definition of densely regularity is needed.

Definition 3.2. *We say that a pseudo-Jacobian $\partial^* X$ of a vector field $X : M \to TM$ is densely regular on M if there exists a dense subset $K \subseteq M$ such that*
(i) $\partial^* X(x)$ is regular at every $x \in K$,
(ii) the pseudo-Jacobian $\partial^* X(x)$ at every $x \notin K$ is contained in the set Ω defined as follows

$$\Omega = \{\lim_{k\to\infty} A_k : \ A_k \in \partial^* X(x_k), \{x_k\} \subset K \ \textit{and} \ x_k \to x\}. \tag{3.7}$$

The following theorem presents a characterization of monotone vector fields in terms of the pseudo-Jacobian.

Theorem 3.3. *Let $X : M \to TM$ be a continuous vector field with a pseudo-Jacobian $\partial^* X(x)$ for each $x \in M$. Assume that for every $x \in M$, all elements of $\partial^* X(x)$ are positive semidefinite, then X is monotone.*

Conversely, if X is monotone and $\partial^* X$ is densely regular on M, then for each $x \in M$, every $A \in \partial^* X(x)$ is positive semidefinite.

Proof. Let $x, y \in M$ and $v \in T_x M$. Since M is complete, there exists a minimizing geodesic $\gamma : [0,1] \to M$ connecting x to y. Consider the real-valued function $g : [0,1] \to \mathbb{R}$ defined by

$$g(t) := \langle v, L_{\gamma(t)x,\gamma} X(\gamma(t)) - X(x) - t(L_{yx,\gamma} X(y) - X(x))\rangle_x,$$

where $L_{xy,\gamma}$ is the parallel transport along γ. Then g is continuous on $[0,1]$ with $g(0) = g(1) = 0$. Hence g attains its minimum and maximum on $[0,1]$.

If both maximum and minimum is attained on the boundary points, then g is equal to zero. Therefore $g^+(t,\alpha) = 0$ for all $t \in (0,1)$ and $\alpha > 0$. Otherwise without loss of generality we can assume that g attains its minimum at at some $t_0 \in (0,1)$. Then for every $\alpha \in \mathbb{R}$,

$$g^+(t_0,\alpha) \geq 0. \tag{3.8}$$

Hence,

$$\begin{aligned} g^+(t_0,\alpha) &= \limsup_{t\downarrow 0} \frac{g(t_0+t\alpha)-g(t_0)}{t} \\ &= \limsup_{t\downarrow 0} \frac{\langle v, L_{\gamma(t_0+t\alpha)x,\gamma} X(\gamma(t_0+t\alpha)) - L_{\gamma(t_0)x,\gamma} X(\gamma(t_0))\rangle_x}{t} \\ &- \alpha\langle v, L_{yx,\gamma}X(y) - X(x)\rangle_x. \end{aligned}$$

We define $\theta(t) := \gamma(t_0 + t\alpha)$. Then $\theta(0) = \gamma(t_0)$ and $\theta'(0) = \alpha\gamma'(t_0)$. Therefore

$$\begin{aligned} g^+(t_0,\alpha) &= \limsup_{t\downarrow 0} \frac{\langle v, L_{\theta(t)x,\gamma} X(\theta(t)) - L_{\theta(0)x,\gamma} X(\theta(0))\rangle_x}{t} \\ &- \alpha\langle v, L_{yx,\gamma}X(y) - X(x)\rangle_x \\ &= \limsup_{t\downarrow 0} \frac{\langle L_{x\theta(0),\gamma} v, L_{x\theta(0),\gamma} L_{\theta(t)x,\gamma} X(\theta(t)) - L_{x\theta(0),\gamma} L_{\theta(0)x,\gamma} X(\theta(0))\rangle_{\theta(0)}}{t} \\ &- \alpha\langle v, L_{yx,\gamma}X(y) - X(x)\rangle_x \\ &= \limsup_{t\downarrow 0} \frac{\langle L_{x\theta(0),\gamma} v, L_{\theta(t)\theta(0),\gamma} X(\theta(t)) - X(\theta(0))\rangle_{\theta(0)}}{t} \\ &- \alpha\langle v, L_{yx,\gamma}X(y) - X(x)\rangle_x \geq 0. \end{aligned}$$

Note that γ is a minimal geodesic connecting x and y, therefore for t small enough θ is the unique minimal geodesic connecting $\theta(0)$ and $\theta(t)$. Hence if we define $\tilde{v} := L_{x\gamma(t_0),\gamma}(v)$,

$$(\tilde{v}X)^+(\gamma(t_0), \alpha\gamma'(t_0)) \geq \alpha\langle v, L_{yx,\gamma}X(y) - X(x)\rangle_x.$$

Now, by taking $\alpha = 1$ and $\alpha = -1$,

$$-(\tilde{v}X)^+(\gamma(t_0), -\gamma'(t_0)) \leq \langle v, L_{yx,\gamma}X(y) - X(x)\rangle_x \leq (\tilde{v}X)^+(\gamma(t_0), \gamma'(t_0)).$$

By the definition of pseudo-Jacobian

$$\inf_{A\in\partial^* X(\gamma(t_0))} \langle \tilde{v}, A\gamma'(t_0)\rangle_{\gamma(t_0)} \leq \langle v, L_{yx,\gamma}X(y) - X(x)\rangle_x \leq \sup_{A\in\partial^* X(\gamma(t_0))} \langle \tilde{v}, A\gamma'(t_0)\rangle_{\gamma(t_0)}.$$

Assume that $v = \gamma'(0)$, therefore $\tilde{v} = \gamma'(t_0)$. Since $A \in \partial^* X(\gamma(t_0))$ is positive semidefinite, we conclude

$$0 \leq \inf_{A\in\partial^* X(\gamma(t_0))} \langle \tilde{v}, A\tilde{v}\rangle_{\gamma(t_0)} \leq \langle \gamma'(0), L_{yx,\gamma}X(y) - X(x)\rangle_x. \tag{3.9}$$

which shows the monotoncity of X.

To prove the converse statement, suppose on the contrary that

$$\langle A_0 u_0, u_0 \rangle_{x_0} < 0,$$

for some $x_0 \in M, u_0 \in T_{x_0}M$ and $A_0 \in \partial^* X(x_0)$. If $x_0 \in K$, then by the regularity condition

$$(u_0 X)^-(x_0, u_0) = \inf_{A \in \partial^* X(x_0)} \langle A u_0, u_0 \rangle_{x_0} < 0.$$

Then there exists $t > 0$ small enough such that

$$\langle u_0, L_{\gamma(t)x_0} X(\gamma(t)) - X(x_0) \rangle_{x_0} < 0,$$

where γ is a geodesic starting at x_0 with $\gamma'(0) = u_0$. This contradicts the monotonicity of the vector field X.

Now assume that $x_0 \notin K$, then we can find a sequence $\{x_n\} \subset K, x_n \to x_0$ and $A_n \in \partial^* X(x_n)$ such that $A_n \to A_0$. Therefore $x_n \to x_0$ and

$$\langle A_n u(x_n), u(x_n) \rangle_{x_n} \longrightarrow \langle A_0 u(x_0), u(x_0) \rangle_{x_0},$$

where $u(x) = L_{x_0 x}(u_0)$. Hence for every $\varepsilon > 0$, there exists N such that

$$\langle A_n u(x_n), u(x_n) \rangle_{x_n} < \langle A_0 u_0, u_0 \rangle_{x_0} + \varepsilon \qquad \forall\, n > N.$$

Assume that $\varepsilon = -\frac{\langle A_0 u_0, u_0 \rangle_{x_0}}{2}$, then $\langle A_n u(x_n), u(x_n) \rangle_{x_n} < 0$. By the regularity condition

$$\begin{aligned} (u(x_n)X)^-(x_n, u(x_n)) &= \inf_{M \in \partial^* X(x_n)} \langle M u(x_n), u(x_n) \rangle_{x_n} \\ &< \langle A_n u(x_n), u(x_n) \rangle_{x_n} < 0. \end{aligned}$$

Therefore for sufficiently small $t > 0$,

$$\langle u(x_n), L_{\tilde{\gamma}(t)x_n} X(\tilde{\gamma}(t)) - X(x_n) \rangle_{x_n} < 0,$$

where $\tilde{\gamma}(0) = x_n, \tilde{\gamma}'(0) = u(x_n)$. This again contradicts the monotonicity of X, and so the proof is complete. □

Theorem 3.4. *Let $X : M \to TM$ be a continuous vector field with a bounded pseudo-Jacobian $\partial^* X(x)$ for each $x \in M$. If every $A \in \partial^* X(x)$ is positive definite for each $x \in M$, then X is strictly monotone.*

Proof. The result follows from the proof of Theorem 3.3 and positive definiteness of pseudo-Jacobian of X at every $x \in M$. □

Theorem 3.5. *Let $X : M \to TM$ be a continuous vector field that admits a pseudo-Jacobian $\partial^* X(x)$ for each $x \in M$. If for each $x \in M$, $A \in \partial^* X(x)$ is antisymmetric, then X is trivially monotone. Conversely, if X is trivially monotone and if pseudo-Jacobian $\partial^* X(x)$ is densely regular on M, then for each $x \in M$, $A \in \partial^* X(x)$ is antisymmetric.*

Proof. By Theorem 3.3, we have that

$$0 = \inf_{A \in \partial^* X(\gamma(t_0))} \langle \tilde{v}, A\tilde{v} \rangle_x \leq \langle v, L_{yx,\gamma}(X(y)) - X(x) \rangle_x \leq \sup_{A \in \partial^* X(\gamma(t_0))} \langle \tilde{v}, A\tilde{v} \rangle_x = 0.$$

Hence $\langle \gamma'(0), L_{yx,\gamma}(X(y)) - X(x)\rangle_x = 0$, which shows that X is trivially monotone.

Conversely, if X is trivially monotone, then X and $-X$ are monotone. The monotonicity of X, densely regularity of $\partial^* X(x)$ and Theorem 3.3 show that for each $x \in M$, $A \in \partial^* X(x)$ is positive semidefinite. Therefore $\langle v, Av\rangle_x \geq 0$ for every $v \in T_x M$. The same argument holds for $-X$ and hence for each $x \in M$ and $\tilde{A} \in \partial^*(-X)(x)$, $\langle v, \tilde{A}v\rangle_x \geq 0$. By Theorem 2.6, $-\partial^* X(x)$ is a pseudo-Jacobian of $-X$ at x, therefore for every $A \in \partial^* X(x)$, we have $-A \in \partial^*(-X)(x)$ and this completes the proof. □

As special cases of Theorem 3.3, we see that if X is locally Lipschitz, then monotonicity of X can be characterized by positive semidefinite Clarke generalized Jacobian, and if X is a smooth vector field, then monotonicity of X is characterized by positive semidefinite linear operator $DX(x)$.

The following corollary characterizes the monotoncity of vector fields in terms of their Clarke generalized Jacobian. The proof can be obtained along the same lines as [10, Corollary 5.1.4].

Corollary 3.6. *Let $X : M \to TM$ be a locally Lipschitz vector field. Then X is monotone if and only if for each $x \in M$, the Clarke generalized Jacobian $A \in \partial_c X(x)$ are positive semidefinite. Moreover, for every $x \in M$, if the Clarke generalized Jacobian $A \in \partial_c X(x)$ are positive definite, then X is strictly monotone on M. Furtheremore, X is trivially monotone if and only if for each $x \in M$, the Clarke generalized Jacobian $A \in \partial_c X(x)$ are antisymmetric.*

Corollary 3.7. *Let $X : M \to TM$ be a smooth vector field. Then*

(i) X is monotone if and only if for each $x \in M$, the linear operators $DX(x)$ are positive semidefinite.

(ii) If for each $x \in M$, the linear operators $DX(x)$ are positive definite, then X is strictly monotone.

(iii) X is trivially monotone if and only if for each $x \in M$, the linear operators $DX(x)$ are antisymmetric.

Convexity plays a central role in mathematical economics, engineering, management science, and optimization theory. Therefore the research on convexity and generalized convexity and their characterizations is one of the most important aspects of mathematical programming. We now proceed to present a second-order characterization of convex functions on Riemannian manifolds. Recall that a function $f : M \to \mathbb{R}$ is called convex (strictly convex), if $f \circ \gamma : [0, 1] \to \mathbb{R}$ for every geodesic γ of M is convex (strictly convex). The following proposition can be found in [5].

Proposition 3.8. *Let $f : M \to \mathbb{R}$ be a smooth function on a Riemannian manifold M.*
(i) The function f is convex if and only if the vector field ∇f is monotone.
(ii) The function f is strictly convex if and only if the vector field ∇f is strictly monotone.

A second-order characterization of convex function can be obtained from the first-order characterization of monotone vector fields.

Corollary 3.9. *Let $f : M \to \mathbb{R}$ be a C^1-function which admits a pseudo-Hessian $\partial^2_* f(x)$ at each $x \in M$. If every $A \in \partial^2_* f(x)$ is positive semidefinite, then f is convex on M. Conversely, If f is convex and the pseudo-Hessian $\partial^2_* f$ is a densely regular pseudo-Jacobian of ∇f on M, then for each $x \in M$, $A \in \partial^2_* f(x)$ are positive semidefinite. Moreover, if for every $x \in M$, the pseudo-Hessian $\partial^2_* f(x)$ are positive definite, then f is strictly convex function on M.*

Corollary 3.10. *Let $f : M \to \mathbb{R}$ be $C^{1,1}$. Then f is convex if and only if for each $x \in M$, every $A \in \partial^2_* f(x)$ is positive semidefinite.*

Now, we characterize quasimonotonicity of continuous vector fields using pseudo-Jacobian.

Theorem 3.11. *Let $X : M \to TM$ be a continuous vector field that admits a pseudo-Jacobian $\partial^* X(x)$ for each $x \in M$.*
(a) If X is quasimonotone, then for every $x \in M$, $u \in T_xM$ and geodesic γ starting from x, we have
(i) $\langle X(x), u\rangle_x = 0$ implies $\sup_{A\in\partial^* X(x)}\langle Au, u\rangle_x \geq 0$.
(ii) $\langle X(x), u\rangle_x = 0$ and $\langle L_{\gamma(\bar t)x,\gamma} X(\gamma(\bar t)), u\rangle_x > 0$ for some $\bar t < 0$ imply the existence of $\tilde t > 0$ such that $\langle L_{\gamma(t)x,\gamma} X(\gamma(t)), u\rangle_x \geq 0$ for all $t \in [0, \tilde t]$.
(b) If X admits a bounded and densely regular pseudo-Jacobian $\partial^* X$ on M and the following conditions hold for every $x \in M, u \in T_xM$, and geodesic γ starting from x,
(i) $\langle X(x), u\rangle_x = 0$ implies $\max_{A\in\partial^* X(x)}\langle Au, u\rangle_x \geq 0$.
(ii) $\langle X(x), u\rangle_x = 0$, $0 \in \{\langle Au, u\rangle_x : A \in \partial^* X(x)\}$ and $\langle L_{\gamma(\bar t)x,\gamma} X(\gamma(\bar t)), u\rangle_x > 0$ for some $\bar t < 0$ imply the existence of $\tilde t > 0$ such that $\langle L_{\gamma(t)x,\gamma} X(\gamma(t)), u\rangle_x \geq 0$ for all $t \in [0, \tilde t]$.
Then X is quasimonotone.

Proof. (a)-(i) Assume on the contrary that there exist $x \in M$, $u \in T_xM$ such that $\langle X(x), u\rangle_x = 0$ and $\sup_{A\in\partial^* X(x)}\langle Au, u\rangle_x < 0$. Therefore by definition we have

$$(uX)^+(x, u) \leq \sup_{A\in\partial^* X(x)}\langle Au, u\rangle_x < 0, \quad (-uX)^+(x, -u) \leq \sup_{A\in\partial^* X(x)}\langle Au, u\rangle_x < 0,$$

which imply that for sufficiently small $t > 0$,

$$\langle u, L_{\gamma(t)x} X(\gamma(t)) - X(x)\rangle_x < 0, \quad \langle -u, L_{\tilde\gamma(t)x} X(\tilde\gamma(t)) - X(x)\rangle_x < 0,$$

where γ and $\tilde\gamma$ are geodesics with $\gamma(0) = x$, $\gamma'(0) = u$ and $\tilde\gamma(0) = x$, $\tilde\gamma'(0) = -u$. Since γ is the unique geodesic connecting x and $\gamma(t)$ for t small enough, and $L_{x\gamma(t)}$ is the parallel transportation mapping along γ, therefore $L_{x\gamma(t)}(u) = \gamma'(t)$ and similarly we have $L_{x\tilde\gamma(t)}(-u) = \tilde\gamma'(t)$ for t small enough. Hence

$$\langle \gamma'(t), X(\gamma(t))\rangle_{\gamma(t)} < 0, \quad \langle \tilde\gamma'(t), X(\tilde\gamma(t))\rangle_{\tilde\gamma(t)} < 0. \tag{3.10}$$

Now, we define $\alpha(s) = \gamma((1-s)t - st)$, then $\alpha(0) = \gamma(t), \alpha(1) = \gamma(-t)$ and $\alpha'(0) = -2t\gamma'(t), \alpha'(1) = -2t\gamma'(-t)$. Without loss of generality, we assume that $\tilde{\gamma}(t) = \gamma(-t)$ for t small enough. Hence the inequality (3.10) implies

$$\langle X(\alpha(0)), \alpha'(0)\rangle_{\alpha(0)} > 0, \quad \langle X(\alpha(1)), \alpha'(1)\rangle_{\alpha(1)} < 0.$$

which is a contradiction and the proof of (i) is complete.

Now we prove (a)-(ii) by contradiction, assume that (ii) does not hold, then for every $\tilde{t}$, there exists $t_0 \in [0, \tilde{t}]$ such that $\langle L_{\gamma(t_0)x,\gamma} X(\gamma(t_0)), u\rangle_x < 0$ while $\langle X(x), u\rangle_x = 0$ and $\langle L_{\gamma(\bar{t})x,\gamma} X(\gamma(\bar{t})), u\rangle_x > 0$ for some $\bar{t} < 0$, where $\gamma(0) = x$ and $\gamma'(0) = u$. Thus

$$\langle X(\gamma(t_0)), \gamma'(t_0)\rangle_{\gamma(t_0)} < 0, \quad \langle X(\gamma(\bar{t})), \gamma'(\bar{t})\rangle_{\gamma(\bar{t})} > 0. \tag{3.11}$$

Let $\alpha : [0,1] \to M$ be defined by $\alpha(t) = \gamma(-(t_0 - \bar{t})t + t_0)$. Then we have $\alpha(0) = \gamma(t_0), \alpha(1) = \gamma(\bar{t})$ and $\alpha'(0) = -(t_0 - \bar{t})\gamma'(t_0)$, $\alpha'(1) = -(t_0 - \bar{t})\gamma'(\bar{t})$. Since $\bar{t} < 0 \le t_0$, therefore inequality (3.11) shows

$$\langle X(\alpha(0)), \alpha'(0)\rangle_{\alpha(0)} > 0, \quad \langle X(\alpha(1)), \alpha'(1)\rangle_{\alpha(1)} < 0.$$

which contradicts the quasimonotonicity of X.

To prove (b), we assume that X is not quasimonotone and there exist $x,\ y \in M$ and geodesic γ joining $x,\ y$ such that

$$\langle X(x), \gamma'(0)\rangle_x > 0 \qquad \text{and} \qquad \langle X(y), \gamma'(1)\rangle_y < 0. \tag{3.12}$$

Since M is complete, the function $g(t) := \langle X(\gamma(t)), \gamma'(t)\rangle_{\gamma(t)}$ is continuous on $\mathbb{R}$ and $g(0) > 0$ and $g(1) < 0$, therefore there exists $t_1 \in (0,1)$ such that

$$g(t_1) = 0 \qquad \text{and} \qquad g(t) < 0 \qquad \text{for all} \qquad t \in (t_1, 1). \tag{3.13}$$

We assume that $x_1 = \gamma(t_1)$ and $\tilde{u} := \gamma'(t_1)$. Then we have that $g(t_1) = \langle X(x_1), \gamma'(t_1)\rangle_{x_1} = 0$ and

$$(\tilde{u}X)^-(x_1, \tilde{u}) = \liminf_{t \downarrow 0} \frac{\langle \tilde{u}, L_{\gamma(t+t_1)x_1} X(\gamma(t+t_1)) - X(x_1)\rangle_{x_1}}{t} \le 0.$$

We claim that $0 \in \{\langle \tilde{u}, A\tilde{u}\rangle_{x_1} : A \in \partial^* X(x_1)\}$. To prove our claim; first assume that $x_1 \in K_0$. The case $\langle \tilde{u}, A\tilde{u}\rangle_{x_1} > 0$ for each $A \in \partial^* X(x_1)$ contradicts the regularity of $\partial^* X(x_1)$. If $\langle \tilde{u}, A\tilde{u}\rangle_{x_1} < 0$ for each $A \in \partial^* X(x_1)$ contradicts (i). Hence the claim is proved for every $x_1 \in K$. Moreover, the claim can be proved for $x_1 \notin K_0$ by using the same arguments on the sequences $\{y_k\} \in K_0$ and A_k where $y_k \to x_1$ and $A_k \in \partial^* X(y_k)$ such that $\lim_{k\to\infty} A_k = A$.

Now, by the continuity of g there exists $t' < 0$ such that

$$g(t_1 + t') = \langle X(\gamma(t_1 + t')), \gamma'(t_1 + t')\rangle_{\gamma(t_1+t')} > 0.$$

By condition (ii), there exists $t_0 > 0$ such that

$$g(t_1 + t) = \langle X(\gamma(t_1 + t)), \gamma'(t_1 + t)\rangle_{\gamma(t_1+t)} \ge 0 \qquad \text{for all } t \in [0, t_0].$$

This contradicts the condition that $g(t) < 0$ for all $t \in (t_1, 1)$. Therefore X is quasimonotone. □

The following corollaries can be proved easily.

Corollary 3.12. *Suppose that X is a locally Lipschitz vector field on a Riemannian manifold M. Then X is quasimonotone if and only if the following conditions hold for each $x \in M,\ u \in T_xM$ and every geodesic γ starting at x:*
(i) $\langle X(x), u\rangle_x = 0$ implies $\max_{A\in\partial_c X(x)}\langle u, Au\rangle_x \geq 0$.
(ii) $\langle X(x), u\rangle_x = 0,\ 0 \in \{\langle u, Au\rangle_x : A \in \partial_c X(x)\}$ and $\langle L_{\gamma(t')x,\gamma} X(\gamma(t')), u\rangle_x > 0$ for some $t' < 0$ imply the existence of $t_0 > 0$ such that $\langle L_{\gamma(t)x,\gamma} X(\gamma(t)), u\rangle_x \geq 0$ for all $t \in [0, t_0]$.

Corollary 3.13. *Suppose that X is a smooth vector field on a Riemannian manifold M. Then X is quasimonotone if and only if the following conditions hold for each $x \in S,\ u \in T_xM$ and every geodesic γ starting at x:*
(i) $\langle X(x), u\rangle_x = 0$ implies $\langle u, DX(x)u\rangle_x \geq 0$.
(ii) $\langle X(x), u\rangle_x = \langle u, DX(x)u\rangle_x = 0$ and $\langle L_{\gamma(t')x,\gamma} X(\gamma(t')), u\rangle_x > 0$ for some $t' < 0$ imply the existence of $t_0 > 0$ such that $\langle L_{\gamma(t)x,\gamma} X(\gamma(t)), u\rangle_x \geq 0$ for all $t \in [0, t_0]$.

Finally, we characterize pseudomonotonicity of a continuous vector field defined on a Riemmannian manifold in terms of its pseudo-Jacobian.

Theorem 3.14. *Let $X : M \to TM$ be a continuous vector field that admits a pseudo-Jacobian $\partial^* X(x)$ for each $x \in M$.*
(a) If X is pseudomonotone, then for every $x \in M$, $u \in T_xM$ and geodesic γ starting at x, $\langle X(x), u\rangle_x = 0$ implies that
(i) $\sup_{A\in\partial^* X(x)}\langle Au, u\rangle_x \geq 0$.
(ii)There exists $t_0 > 0$, such that $\langle L_{\gamma(t)x,\gamma} X(\gamma(t)), u\rangle_x \geq 0$ for all $t \in [0, t_0]$.
(b) Suppose that the following conditions hold for every $x \in M, u \in T_xM$, and geodesic γ starting at x:
(i) $\langle X(x), u\rangle_x = 0$ implies $\max_{A\in\partial^* X(x)}\langle Au, u\rangle_x \geq 0$.
(ii) $\langle X(x), u\rangle_x = 0$ and $0 \in \{\langle Au, u\rangle_x : A \in \partial^* X(x)\}$ imply the existence of $t_0 > 0$ such that $\langle L_{\gamma(t)x,\gamma} X(\gamma(t)), u\rangle_x \geq 0$ for all $t \in [0, t_0]$.
Then X is pseudomonotone.

Proof. pseudomonotonicity implies quasimonotonicity therefore (i)-(a) follows from (3.11). If (ii)-(a) does not hold, then there exist $x \in M$ and $t' > 0$ such that $\langle X(x), u\rangle = 0$ and $\langle L_{\gamma(t')x,\gamma} X(\gamma(t')), u\rangle_x < 0$, where $\gamma(0) = x,\ \gamma'(0) = u$. Let $y = \gamma(t')$ and $\alpha : [0, 1] \to M$ be the geodesic arc defined by $\alpha(t) = \gamma(-t't + t')$. Therefore we have $\alpha(0) = \gamma(t'),\ \alpha(1) = \gamma(0)$ and $\alpha'(0) = -t'\gamma'(t')$, $\alpha'(1) = -t'\gamma'(0)$. Therefore

$$\langle X(x), u\rangle_x = \langle X(\alpha(1)), \alpha'(1)\rangle_x = 0, \quad \langle X(\alpha(0)), \alpha'(0)\rangle_{\alpha(0)} > 0. \tag{3.14}$$

Now it follows from pseudomonotonicity that $\langle X(\alpha(1)), \alpha'(1)\rangle_{\alpha(1)} > 0$, which is a contradiction.

To prove (b), suppose that there exist $x,\ y \in M$ and a geodesic γ joining $x,\ y$ such that

$$\langle X(x), \gamma'(0)\rangle_x \geq 0 \qquad \text{and} \qquad \langle X(y), \gamma'(1)\rangle_y < 0. \tag{3.15}$$

Define $g(t) = \langle X(\gamma(t)), \gamma'(t)\rangle_{\gamma(t)}$. Since g is continuous, $g(0) \geq 0$ and $g(1) < 0$, therefore there exists $t_1 \in [0, 1]$ such that $g(t_1) = 0$ and $g(t) < 0$ for all $t \in (t_1, 1]$. Define $x_1 = \gamma(t_1)$ and $\tilde{u} := L_{x\gamma(t_1)}(u)$. We conclude along the same lines as in the proof of Theorem 3.11 that $\langle X(x_1), \tilde{u}\rangle_{x_1} = 0,\ (\tilde{u}X)^-(x_1, \tilde{u}) \leq 0$ and $0 \in \{\langle \tilde{u}, A\tilde{u}\rangle_{x_1} : A \in \partial^* X(x_1)\}$. Now it follows from (ii) that there exists $t_0 > 0$ such that

$$g(t_1 + t) = \langle X(\gamma(t_1 + t)), \gamma'(t_1 + t)\rangle_{\gamma(t_1+t)} \geq 0 \qquad \text{for all } t \in [0, t_0]. \tag{3.16}$$

This contradicts the condition that $g(t) < 0$ for all $t \in (t_1, 1)$. Hence X is pseudomonotone. □

Corollary 3.15. *Suppose that X is a locally Lipschitz vector field on a Riemannian manifold M. Then X is pseudomonotone if and only if the following conditions hold for each $x \in M,\ u \in T_xM$ and every geodesic γ starting at x:*
(i) $\langle X(x), u\rangle_x = 0$ implies $\max_{A \in \partial_c X(x)} \langle u, Au\rangle_x \geq 0$.
(ii) $\langle X(x), u\rangle_x = 0$ and $0 \in \{\langle u, Au\rangle_x : A \in \partial_c X(x)\}$ imply the existence of $t_0 > 0$ such that $\langle L_{\gamma(t)x,\gamma} X(\gamma(t)), u\rangle_x \geq 0$ for all $t \in [0, t_0]$.

Corollary 3.16. *Suppose that X is a smooth vector field on a Riemannian manifold M. Then X is pseudomonotone if and only if the following conditions hold for each $x \in M,\ u \in T_xM$ and every geodesic γ starting at x:*
(i) $\langle X(x), u\rangle_x = 0$ implies $\langle u, DX(x)u\rangle_x \geq 0$.
(ii) $\langle X(x), u\rangle_x = \langle u, DX(x)u\rangle_x = 0$ imply the existence of $t_0 > 0$ such that $\langle L_{\gamma(t)x,\gamma} X(\gamma(t)), u\rangle_x \geq 0$ for all $t \in [0, t_0]$.

We recall that for each type of generalized monotonicity, there is a corresponding type of generalized convexity. Therefore we have the following proposition deduced from the Theorem 3.11 and this fact that a gradient vector field ∇f is quasimonotone if and only if, the function f is quasiconvex. Note that a function $f : M \to \mathbb{R}$ is called quasiconvex on a Riemannian manifold M if, for all $x,\ y \in M,\ x \neq y,\ t \in [0, 1]$ and every geodesic arc γ joining x and y, we have that

$$f(y) \leq f(x) \qquad \text{implies} \qquad f(\gamma(t)) \leq f(x).$$

Proposition 3.17. *Let $f : M \to \mathbb{R}$ be a C^1-function that admits a pseudo-Hessian $\partial^2_* f(x)$ at each $x \in M$. If f is quasiconvex, then for each $x \in M$ and $u \in T_xM$ with $\langle \nabla f(x), u\rangle_x = 0$,*

$$\sup_{A \in \partial^2_* f(x)} \langle Au, u\rangle_x \geq 0.$$

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