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CHARACTERIZATION OF LOWER SEMICONTINUOUS CONVEX FUNCTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, an upper subderivative of a lower semicontinuous function on a Riemannian manifold is introduced. Then, an approximate mean value theorem for the upper subderivative on a Hadamard manifold is presented. Moreover, the results are used for characterization of convex functions on Riemannian manifolds.

1. INTRODUCTION

This paper is concerned with characterization of lower semicontinuous convex functions on Hadamard manifolds via the monotonicity of their subdifferentials. Subdifferential calculus was started with convex functions on $\mathbb{R}^n$. Rockafellar defined the subdifferential for such functions and characterized it in terms of one-sided directional derivatives; see [26]. F. H. Clarke in his thesis demonstrated how the definition of the subdifferential could be extended to arbitrary lower semicontinuous functions in such a way that it is the same as the subdifferential of convex analysis when the function is convex. He also characterized the subdifferential of a locally Lipschitz function by a generalized directional derivative; see [15]. In [27], Rockafellar replaced the class of locally Lipschitz functions by classes of noncontinuous functions and characterized the subdifferential in terms of a directional derivative called the upper subderivative. In [29], a mean value theorem for a lower semicontinuous function on a Banach space using the upper subderivative defined by Rockafellar, was proved. This mean value theorem has many applications in sufficient optimality conditions and characterizations of convex lower semicontinuous functions; see [16, 29].

Convexity of functions and sets plays an important role in economics, management science, and mathematical theories, etc. Therefore, the study of convex functions and other concepts related to convexity are essential from both the theoretical and practical points of view. First-order characterizations for the convexity of extended real-valued functions via the monotonicity of the Fréchet derivative and the monotonicity of the Fréchet subdifferential mapping or the limiting subdifferential mapping can be found, e.g., in [19] and [24, Theorem 3.56]. The convexity can be characterized also by using first-order directional derivatives; see e.g. [19] and the references therein.

Key words and phrases. Riemannian manifolds, Subdifferential, Lower semicontinuous functions, Locally Lipschitz functions.

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Extension of concepts of nonsmooth analysis to Riemannian manifolds are necessary due to the fact that nonsmooth functions on manifolds have a lot of applications, such as in computer vision, signal processing, motion and structure estimation; see [2, 3]. Some other works and applications include those by Absil, Mahony and Sepulchre [1], Azagra, Ferrera, López-Mesas, Sanz [6, 7, 8, 9] and references therein. Furthermore, some attempts have been made to develop the concepts of convex functions and monotone vector fields on Riemannian manifolds; [10, 25].

In [20], a notion of the Clarke subdifferential for locally Lipschitz functions defined on Riemannian manifolds was introduced, a calculus for nonsmooth functions on these manifolds was established and its applications were discussed. These motivated us to replace the class of locally Lipschitz functions on manifolds by lower semicontinuous functions to develop the results in [16, 27, 29] to manifold settings.

This paper is devoted to the study of the Rockafellar subdifferential for lower semicontinuous functions defined on Riemannian manifolds. We develop a basic calculus result for this subdifferential. Then, we prove an approximate mean value theorem for the upper subderivative on Hadamard manifolds. It is worthwhile to mention that our proof is based on the convexity of the distance function on manifolds, hence we have to work on Hadamard manifolds. Moreover, a characterization of lower semicontinuous convex functions on manifolds by the monotonicity of its subdifferential is presented.

In this paper, we use the standard notations and known results of Riemannian manifolds, see, e.g. [28].

Throughout this paper M is a finite dimensional manifold endowed with a Riemannian metric $g_x(.,.)$ on the tangent space T_xM[1]. A Riemannian metric is therefore a smooth assignment of an inner product to each tangent space. It is usual to write

$$g_x(v,w) = \langle v,w\rangle_x, \quad \text{for all } v,w \in T_xM.$$

A Riemannian metric allows us to compute the length of any vector (as well as the angle between two vectors with the same base point). Let us recall the length of a piecewise C^1 curve.[2]

Definition 1.1. *Let $\gamma : [a,b] \to M$ be a piecewise C^1 curve, then its length is defined by*

$$L(\gamma) := \int_a^b \|\gamma'(t)\| dt = \int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}} dt.$$

For two points x, $y \in M$, we define

$$d(x,y) := \inf\{L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve joining } x \text{ to } y\}.$$

Then, d is a distance (called $g-$distance) on M, which defines the same topology as the one M naturally has as a manifold.

The tangent bundle of a manifold M is the disjoint union of the tangent spaces of M, i.e., $TM = \sqcup_{x\in M} T_xM$, where T_xM denotes the tangent space to M at the point x.

Given a manifold M, a vector field on M is an assignment of a tangent vector to each point in M. More precisely, a vector field F is a mapping from M into the tangent bundle TM such that $\pi \circ F$ is the identity mapping, where π denotes the

[1] $T_xM = \{v | \; \exists\gamma : (-\epsilon,\epsilon) \to M, \; \gamma(0) = x, \; \gamma'(0) = v \; \}$.

[2] A function f is said to be of class C^k if the derivatives $f', f'', ..., f^{(k)}$ exist and are continuous.

projection map, i.e, $\pi : TM \to M$ is defined by $\pi(x, v) := x$. The collection of all smooth vector fields is also denoted by $\mathfrak{X}(M)$.

Let M be a Riemannian manifold. Then an affine connection[3] is called a Levi-Civita connection if
(1) it preserves the metric, i.e., $\nabla g = 0$.
(2) it is torsion-free, i.e., $\nabla_X Y - \nabla_Y X = XY - YX$, for any vector fields X and Y in $\mathfrak{X}(M)$.

On any Riemannian manifold M there exists a unique Levi-Civita connection. A vector field defined along a differentiable curve $\gamma : I \to M$ is a differentiable map $V : I \to TM$ such that $V(t) \in T_{\gamma(t)}M$ for all $t \in I$. A vector field V defined along a differentiable curve $\gamma : I \to M$ is called parallel when $\nabla_{\gamma'(t)} V = 0$ for all $t \in I$. We also recall that a geodesic is a C^∞ smooth curve γ satisfying the equation

$$\nabla_{\gamma'(t)} \gamma'(t) = 0.$$

The existence theorem for ordinary differential equations implies that for every $x \in M$ and $(x, v) \in TM$, there exist an open interval J containing 0 and exactly one geodesic $\gamma_v : J \to M$ with $\gamma_v'(0) = v$ and $\gamma_v(0) = x$. Recall a Riemannian manifold M is called geodesically complete if the maximum interval of the definition of every geodesic in M is $\mathbb{R}$.

The exponential mapping $\exp_x : U \subset T_xM \to M$ is then defined as $\exp_x(v) = \gamma_v(1)$.

We identify (via the Riemannian metric) the tangent space of M at a point x with the cotangent space at x, denoted by T_xM^*. We recall that a simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. Also, let S be a nonempty closed subset of a Riemannian manifold M, we define $d_S : M \longrightarrow \mathbb{R}$ by

$$d_S(x) := \inf\{d(x, s) : s \in S\ \},$$

where d is the Riemannian distance on M.

Recall that the set S in a Riemannian manifold M is called to be convex if every two points $x_1, x_2 \in S$ can be joined by a unique geodesic whose image belongs to S. Also, f defined on a Riemannian manifold M is called to be convex provided that $f \circ \gamma : I \subset \mathbb{R} \to \mathbb{R}$ is convex for every geodesic $\gamma : I \to \mathbb{R}$. It is worth mentioning that convexity depends on Riemannian connection, therefore a function may be convex with respect to one Riemannian connection but not to another; for examples see [21, page 294].

A real-valued function f defined on a Riemannian manifold M is said to satisfy a Lipschitz condition of rank K on a given subset S of M if $\mid f(x) - f(y) \mid \leq K d(x, y)$ for every $x, y \in S$, where d is the Riemannian distance on M. A function f is said to be Lipschitz near $x \in M$, if it satisfies the Lipschitz condition of some rank on an open neighborhood of x. A function f is said to be locally Lipschitz on M, if f is Lipschitz near x, for every $x \in M$.

[3]An affine connection on M is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

defined by

$$(X, Y) \mapsto \nabla_X Y$$

such that for all smooth real functions f defined on M and all vector fields X, Y on M:
(1)$\nabla_{fX} Y = f \nabla_X Y$,
(2)$\nabla_X fY = X(f)Y + f \nabla_X Y$.

Now we start with some definitions of nonsmooth analysis on Riemannian manifolds; for more details see [4, 6, 8, 20]. The following definition has appeared for the first time in [18]; see also [5].

Definition 1.2. *Let M be a Riemannian manifold, $x \in M$ and $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function. The proximal subdifferential of f at x, denoted by $\partial_P f(x)$, is defined as $\partial_P(f \circ \exp_x)(0_x)$.*

As a consequence of the definition of $\partial_P(f \circ \exp_x)(0_x)$ one has, $\xi \in \partial_P f(x)$ if and only if there is $\sigma > 0$ such that

$$f(y) \geq f(x) + \langle \xi, \exp_x^{-1}(y)\rangle_x - \sigma d(x,y)^2 \tag{1.1}$$

for every y in a neighborhood of x, [20].

The Fenchel subdifferential of a function f defined on a Hadamard manifold M at a point $x \in M$ such that $f(x) \in \mathbb{R}$ is the set

$$\partial^c f(x) := \{v \in T_xM |\ f(\exp_x(d)) - f(x) \geq \langle v, d\rangle_x \text{ for all } d \in T_xM\}.$$

If $x \notin \mathrm{dom}(f)$, then we define $\partial^c f(x) := \emptyset$.

Definition 1.3. *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function. The Fréchet subdifferential of f at a point $x \in \mathrm{dom} f = \{x \in M :\ f(x) < \infty\}$ is defined as the set $\partial_F f(x)$ of all $\xi \in T_xM$ with the property that*

$$\liminf_{v \to 0_{\|v\| \neq 0}} \|v\|^{-1}(f \circ \exp_x(v) - f(x) - \langle \xi, v\rangle_x) \geq 0.$$

One can deduce that $\partial_F f(x) = \partial_F(f \circ \exp_x)(0_x)$.

The following lemma is another interesting characterization of the Fréchet subdifferential. It is the definition of the subdifferential most often present in the literature on viscosity solution of differential equations, and the Fréchet subdifferential is sometimes referred to as the "viscosity subdifferential"; see [14, 6].

Lemma 1.4. *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function defined on a Riemannian manifold M.*

$$\partial_F f(x) = \{D\varphi(x) | \varphi \in C^1(M, \mathbb{R}), f - \varphi \textit{ attains a local minimum at } x\}.$$

Using the previous lemma and by Corollary 2.4 of [7], we can prove that

$$\partial_P f(x) = \{D\varphi(x) | \varphi \in C^2(M, \mathbb{R}), f - \varphi \text{ attains a local minimum at } x\}.$$

Consequently, the following corollary can be proved easily.

Corollary 1.5. *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function defined on a Riemannian manifold M. Then*

$$\partial_P f(x) \subset \partial_F f(x).$$

Note that the expression $y \to_f x$ means that

$$y \to x, f(y) \to f(x).$$

Definition 1.6. *Suppose that $f : M \to (-\infty, +\infty]$ is a lower semicontinuous function on a Riemannian manifold M. Then, the upper subderivative of f at x in the direction $v' \in T_xM$ denoted by $f^\uparrow(x; v')$ is defined as follows*

$$f^\uparrow(x; v') = \limsup_{y \to_f x,\ t \downarrow 0} \inf_{v \to v'} \frac{f \circ \varphi^{-1}(\varphi(y) + tD\varphi(x)(v)) - f \circ \varphi^{-1}(\varphi(y))}{t}, \tag{1.2}$$

where (φ, U) *is a chart at* x.

Indeed, $f^{\uparrow}(x; v') := (f \circ \varphi^{-1})^{\uparrow}(\varphi(x); D\varphi(x)(v'))$. Note that this definition does not depend on charts.

Considering $0_x \in T_xM$, we have

$$f^{\uparrow}(x; v') = (f \circ \exp_x)^{\uparrow}(0_x; v'). \tag{1.3}$$

The subdifferential $\partial f(x)$ of f at $x \in \mathrm{dom}(f)$ is defined by

$$\partial f(x) = \{x^* \in T_xM |\ \langle x^*, v\rangle_x \le f^{\uparrow}(x; v) \text{ for all } v \in T_xM\}.$$

If $x \notin \mathrm{dom}(f)$, then we define $\partial f(x) = \emptyset$.

Therefore, we can deduce that $\partial f(x) = \partial(f \circ \exp_x)(0_x)$.

Lemma 1.7. *Let* $f : M \to (-\infty, +\infty]$ *be a lower semicontinuous function defined on a Hadamard manifold* M. *Then*

$$\partial^c f(x) \subset \partial_P f(x) \subset \partial_F f(x) \subset \partial f(x).$$

Proof. Assume that $\xi \in \partial^c f(x)$, then for every $d \in T_xM$

$$f(\exp_x(d)) - f(x) \ge \langle \xi, d\rangle_x.$$

Moreover, $\exp_x^{-1}$ is diffeomorphism on M. Therefore, for every y in M and $\sigma > 0$,

$$f(y) - f(x) \ge \langle \xi, \exp_x^{-1}(y)\rangle_x \ge \langle \xi, \exp_x^{-1}(y)\rangle_x - \sigma d(x, y)^2,$$

which implies $\xi \in \partial_P f(x)$. By Corollary 1.5, $\partial_P f(x) \subset \partial_F f(x)$. Due to the fact that $\partial f(x) = \partial(f \circ \exp_x)(0_x)$, $\partial_F f(x) = \partial_F(f \circ \exp_x)(0_x)$ and Property 2.7 in [16], we can conclude that $\partial_F f(x) \subset \partial f(x)$. □

For the next lemme see [20, Lemma 4.8].

Lemma 1.8. *If* $f : M \to \mathbb{R}$ *is locally Lipschitz on a Riemannian manifold* M, *then*

$$\partial f(x) = \mathrm{clconv}\{\lim_{i\to\infty} \xi_i :\ \xi_i \in \partial_P f(y_i),\ y_i \to x\},$$

where clconv *denotes the closed convex hull of the set .*

We recall the definition of the Clarke generalized directional derivative; see [6]. It is worth mentioning that an equivalent definition has appeared in [12].

Definition 1.9. *Suppose that* $f : M \to \mathbb{R}$ *is a locally Lipschitz function on a Riemannian manifold* M. *Then, the generalized directional derivative of* f *at* $x \in M$ *in the direction* $v \in T_xM$, *denoted by* $f^\circ(x; v)$, *is defined as*

$$f^\circ(x; v) := \limsup_{y\to x,\ t\downarrow 0} \frac{f \circ \varphi^{-1}(\varphi(y) + tD\varphi(x)(v)) - f \circ \varphi^{-1}(\varphi(y))}{t}, \tag{1.4}$$

where (φ, U) *is a chart at* x.

Indeed, $f^\circ(x; v) := (f \circ \varphi^{-1})^\circ(\varphi(x); D\varphi(x)(v))$.

Considering $0_x \in T_xM$, we have

$$f^\circ(x; v) = (f \circ \exp_x)^\circ(0_x; v). \tag{1.5}$$

The following two theorems can be proved using (1.3), (1.5) and results in [26].

Theorem 1.10. *Let $f : M \to \mathbb{R}$ be locally Lipschitz on a Riemannian manifold M, then the upper subderivative of f reduces to the Clarke generalized directional derivative. Moreover, if f is convex and locally Lipschitz, then the upper subderivative of f at $x \in M$ in the direction of $v \in T_xM$ is equal to*

$$\lim_{t\downarrow 0} \frac{f(\gamma(t)) - f(x)}{t},$$

where γ is a geodesic starting at x with $\gamma'(0) = v$.

Theorem 1.11. *Let $f : M \to (-\infty, +\infty]$ be lower semicontinuous and finite at x.*
(a) The function $v \to f^\uparrow(x; v)$ is positively homogeneous.
(b) One has $\partial f(x) = \emptyset$ if and only if $f^\uparrow(x; 0) = -\infty$, otherwise

$$f^\uparrow(x; v) = \sup\{\langle x^*, v\rangle_x | \ x^* \in \partial f(x)\} \text{ for all } v \in T_xM.$$

(c) If f attains at x a local minimum, then

$$f^\uparrow(x; v) \geq 0 \text{ for all } v \in T_xM,$$

and

$$0 \in \partial f(x).$$

(d) Suppose that g is a real-valued locally Lipschitz function defined on M. Then

$$\partial(f + g)(x) \subset \partial f(x) + \partial g(x),$$

$$(f + g)^\uparrow(x; v) \leq f^\uparrow(x; v) + g^\uparrow(x; v) \text{ for all } v \in T_xM.$$

(e) The set $\{(v, \alpha) \in T_xM \times \mathbb{R}| \ \alpha \geq f^\uparrow(x; v)\}$ is convex and closed.

Example 1.12. Let $\mathbb{S}^n$ be the linear space of real $n \times n$ symmetric matrices and $\mathbb{S}^n_{++}$ be the space of symmetric positive definite real $n \times n$ matrices. We consider the manifold $M := \mathbb{S}^n_{++}$ with a Riemannian metric on the tangent space $T_XM = \mathbb{S}^n$ defined by

$$\langle A, B\rangle_X := \text{trace } (B\phi''(X)A) = \text{trace } (BX^{-1}AX^{-1}),$$

where trace denotes the trace of a matrix, $A, B \in T_XM$, $X \in M$, $\phi(X) := -\ln \det X$ and ϕ'' denotes the Euclidean Hessian of ϕ. We assume that Ω is an open convex subset of M and $I = \{1, \ldots, m\}$. Let $F_i : \Omega \to \mathbb{R}$ be a continuously differentiable function on Ω, for $i \in I$. If we define $F(X) := \max_{i\in I} F_i(X)$, then F is locally Lipschitz on Ω and

$$\partial F(X) = \text{conv}\{\text{grad } F_i(X) : \ i \in I(X)\},$$

where grad $F_i(X)$ denotes the unique vector in T_XM which satisfies

$$\langle \text{grad } F_i(X), \xi\rangle = DF_i(X)(\xi) \quad \text{for all } \xi \in T_XM.$$

and $I(X) = \{i \in I : \ F(X) = F_i(X)\}$; see Lemma A.3 in [13].

Theorem 1.13. *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function on a Riemannian manifold M. Then, ∂f is nonempty in a dense subset of* $\text{dom}(f)$*. Moreover, if f is locally Lipschitz, then* $\text{dom}(\partial f) = M$*.*

Proof. Since the domain of the Fréchet subdifferential of a lower semicontinuous function f defined on a Riemannian manifold M is dense in $\text{dom}(f)$; see [6, Theorem 4.4], and $\partial_F f(x) \subset \partial f(x)$, we can deduce that ∂f is nonempty in a dense subset of $\text{dom}(f)$.

For the second part of the theorem, we assume that x is an arbitrary point in M, by Theorem 10 in [5] there exists a sequence $y_i \in \operatorname{dom}(\partial_P f)$ convergent to x. Therefore, $\partial_P f(y_i) \neq \emptyset$ and there exists some $\xi_i \in \partial_P f(y_i)$. Since f is locally Lipschitz and $y_i \to x$, there exists a subsequence of ξ_i convergent to some ξ. By Lemma 1.8, $\xi \in \partial f(x) \neq \emptyset$. □

2. Approximate mean value Theorem and characterization of convex functions

The purposes of this section are twofold. First, we give a mean value theorem for a lower semicontinuous function f on a Hadamard manifold M. Then, we use this mean value theorem to characterize lower semicontinuous convex functions on Hadamard manifolds. It is worth pointing out that in this mean value theorem the function f is assumed to be lower semicontinuous, however the mean value theorem for the proximal subdifferential proved by Azagra and Ferrera in [5] was for locally Lipschitz functions. At the first step, we need to present some obvious properties of the distance function on a Riemannian manifold.

Lemma 2.1. *Let C be a compact subset of a Riemannian manifold M. If a sequence (x_k) in M satisfies*

$$\lim_{k\to\infty} d_C(x_k) = 0,$$

then there exists a subsequence (x_{k_n}) which converges to $c \in C$.

Proof. Since $\lim_{k\to\infty} d_C(x_k) = 0$, there exists $N \in \mathbb{N}$ such that for $k > N$, $d_C(x_k) < 1/k$. Moreover, there exists $c_k \in C$ such that $d(c_k, x_k) \leq 1/k$. Since C is compact, we can choose a subsequence (c_{k_n}) which converges to some c in C. Furthermore,

$$d(x_{k_n}, c) \leq d(c_{k_n}, x_{k_n}) + d(c_{k_n}, c),$$

which proves that x_{k_n} converges to c. □

In the following lemma, we need to use convexity of the distance function, therefore we assume that M is a Hadamard manifold.

Lemma 2.2. *Let M be a Hadamard manifold, $a,\ b \in M$, and $\gamma : [0, d(a, b)] \to M$ be the unique geodesic connecting a, b. Then, for every $x \in M$*

$$d^{\uparrow}_{\operatorname{Img}(\gamma)}(x; \exp_x^{-1}(b)) \leq -d_{\operatorname{Img}(\gamma)}(x),$$

where $\operatorname{Img}(\gamma)$ *denotes the image of γ in M.*

Proof. Since $\operatorname{Img}(\gamma)$ is convex in M, by Lemma 5.2 of [23] one can deduce that $d_{\operatorname{Img}(\gamma)}$ is convex. Assume that $\sigma : [0, 1] \to M$ is a geodesic connecting x and b, then for every $t \in [0, 1]$,

$$d_{\operatorname{Img}\gamma}(\sigma(t)) \leq (1-t)d_{\operatorname{Img}\gamma}(x) + t d_{\operatorname{Img}\gamma}(b) = (1-t)d_{\operatorname{Img}\gamma}(x).$$

Hence, Theorem 1.10 implies that

$$d^{\uparrow}_{\operatorname{Img}(\gamma)}(x; \exp_x^{-1}(b)) = \lim_{t\downarrow 0} \frac{d_{\operatorname{Img}(\gamma)}(\sigma(t)) - d_{\operatorname{Img}(\gamma)}(x)}{t} \leq -d_{\operatorname{Img}(\gamma)}(x).$$

□

Lemma 2.3. *If $f : M \to (-\infty, +\infty]$ is lower semicontinuous on a Riemannian manifold M, $a, b \in \operatorname{dom}(f)$ and γ is a geodesic connecting a and b, then there is $\varepsilon > 0$ such that f is bounded from below on the set $\{x \in M |\ d_{\operatorname{Img}(\gamma)}(x) \leq \varepsilon\}$.*

The distance function is differentiable at $(x, y) \in M \times M$ if and only if there is a unique length minimizing geodesic from x to y. Furthermore, the distance function is smooth in a neighborhood of (x, y) if and only if x and y are not conjugate points along this minimizing geodesic. Therefore the distance function is nondifferentiable at (x, y) if and only if $x = y$ or x and y are the conjugate points. Let the distance function be differentiable at (x, y), then

$$\frac{\partial d}{\partial x}(x, y) = \frac{-\exp_x^{-1}(y)}{d(x, y)}.$$

In the next lemma we assume that $x = y$, and find a formula for the subdifferential of the distance function.

Lemma 2.4. *Let M be a complete Riemannian manifold. If $d_x : M \to \mathbb{R}$ is defined by $d_x(y) = d(x, y)$, then*

$$\partial d_x(x) = B,$$

where B is the closed unit ball of T_xM.

Before proving the lemma, let us recall a definition of the normal cone and the tangent cone to a closed convex subset of a Riemannian manifold; for more details see [20]. Let S be a closed convex subset of a Riemannian manifold M, the normal cone to S at x denoted by $N_S(x)$ and the tangent cone to S at x denoted by $T_S(x)$ are defined by

$$N_S(x) = \{\xi \in T_xM | \langle \xi, \exp_x^{-1}(y)\rangle_x \leq 0 \text{ for every } y \in S\}.$$
$$T_S(x) := \{\xi \in T_xM | \langle \xi, v\rangle_x \leq 0 \ \forall v \in N_S(x)\}.$$

Assume that $S = \{x\}$, then $N_S(x) = T_xM$.

Proof. Let $M = \mathbb{R}^n$ and S be a closed convex subset of M, we claim that for every $x \in S$, $\partial d_S(x) = N_S(x) \cap B$.

Let $\xi \in N_S(x) \cap B$. For every $y \in M$, there exists $z \in S$ such that $d_S(y) = d(z, y)$. By the definition of the normal cone

$$\langle \xi, y - x\rangle \leq \langle \xi, z - x\rangle + \langle \xi, y - z\rangle \leq 0 + \|\xi\|\|y - z\| \leq d(z, y),$$

which implies $\langle \xi, y - x\rangle \leq d_S(y) - d_S(x)$, and $\xi \in \partial d_S(x)$.

Now assume that $\xi \in \partial d_S(x)$, so by [20, Theorem 4.10] $d^\circ(x, v) \leq 0$ for every $v \in T_S(x)$. Moreover, by the definition of the support function, $\langle \xi, v\rangle_x \leq 0$, which means $\xi \in N_S(x)$.

Now we assume that M is a Riemannian manifold, $S = \{x\}$. First, we prove that $\partial d_x(x) = \partial d^*_{0_x}(0_x)$, where d, d^* are respectively the Riemannian distance on M and the usual distance on T_xM. By Proposition 2.5 in [20], $\xi \in \partial d_x(x)$ if and only if $\xi \in \partial(d_x \circ \exp_x)(0_x)$ if and only if $\langle \xi, v\rangle_x \leq d_x \circ \exp_x(v) - d_x \circ \exp_x(0_x)$, for every $v \in T_xM$, if and only if $\langle \xi, v\rangle_x \leq \|v\|$ for every $v \in T_xM$, which means $\langle \xi, v\rangle_x \leq d^*_{0_x}(v) - d^*_{0_x}(0_x)$ for every $v \in T_xM$, and by the definition of the subdifferential $\xi \in \partial d^*_{0_x}(0_x)$. Hence, $\partial d_x(x) = \partial d^*_{0_x}(0_x)$.

It is worthwhile to mention that by Proposition 4.3 in [20], $N_x(x) = N_{0_x}(0_x)$. Therefore, by the claim,

$$N_x(x) \cap B = N_{0_x}(0_x) \cap B = \partial d^*_{0_x}(0_x) = \partial d_x(x).$$

As it was mentioned before $N_x(x) = T_xM$, so $\partial d_x(x) = B$. $\square$

Now, we can prove the approximate mean value theorem for lower semicontinuous functions on Hadamard manifolds.

Theorem 2.5. *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function on a Hadamard manifold M. Assume that f is finite at a and b, and γ is the unique geodesic connecting these two points. Then for every $b \neq x \in \mathrm{Img}(\gamma)$ with*

$$f(x) + \frac{f(b) - f(a)}{d(a, b)} d(x, b) \leq f(y) + \frac{f(b) - f(a)}{d(b, a)} d(y, b) \qquad \forall y \in \mathrm{Img}(\gamma), \tag{2.1}$$

there exist sequences (x_k) in M and (x^*_k) in $T_{x_k}M$ such that

$$\lim_{k\to\infty} x_k = x. \tag{2.2}$$

$$\limsup_{k\to\infty} f(x_k) \leq f(a) + \frac{f(b) - f(a)}{d(b, a)} d(x, a), \tag{2.3}$$

$$x^*_k \in \partial f(x_k) \text{ for every } k, \tag{2.4}$$

$$\liminf_{k\to\infty} \langle x^*_k, \exp^{-1}_{x_k}(b)\rangle_{x_k} \geq \frac{f(b) - f(a)}{d(b, a)} d(b, x). \tag{2.5}$$

Proof. By Lemma 2.3, there is $\varepsilon > 0$ such that f is bounded from below on the set $A := \{x \in M |\ d_{\mathrm{Img}(\gamma)}(x) \leq \varepsilon\}$. Define a function g on M by

$$g(y) = f(y) + \frac{f(b) - f(a)}{d(b, a)} d(y, b).$$

Assume that $x \in \mathrm{Img}(\gamma)$ satisfies in (2.1). Therefore, $g(x) \leq g(y)$ for all $y \in \mathrm{Img}(\gamma)$. Now, define a sequence of functions on M by

$$g_{j,k}(y) = j d_{\mathrm{Img}(\gamma)}(y) + g(y) + \frac{d(y, x)}{k}.$$

Since $g_{j,k}$ is lower semicontinuous, so is bounded from below on A. By Ekeland variational principle on complete Riemannian manifolds [6], there exists a sequence $(y_{j,k})$ in A such that

$$g_{j,k}(y_{j,k}) \leq g(x) \qquad \text{for all } j,\ k, \tag{2.6}$$

and

$$g_{j,k}(y) - g_{j,k}(y_{j,k}) \geq \frac{d(y, y_{j,k})}{-k} \qquad \text{for all } y \in A \text{ and } j,\ k. \tag{2.7}$$

Note that g is bounded from below on A, so by (2.6)

$$\lim_{j\to\infty} d_{\mathrm{Img}(\gamma)}(y_{j,k}) = 0 \qquad \text{for all } k.$$

Therefore, Lemma (2.1) implies that there exists a subsequence $(y_{j_n,k})$ which converges to $(y_k) \in \mathrm{Img}(\gamma)$. Note that $g(x) \leq g(y_k)$ for all k. Moreover, (2.6) and the lower semicontinuity of g imply that

$$g(y_k) + \frac{d(y_k, x)}{k} \leq g(x) \text{ for all } k.$$

Thus

$$g(x) \leq g(y_k) + \frac{d(y_k, x)}{k} \leq g(x) \text{ for all } k,$$

which shows $y_k = x$. We can assume that $x_k = y_{j_{n_k},k}$ for all k, where j_{n_k} is increasing, then $\lim_{k\to\infty} x_k = x$. Since $g(x) \le g(y)$ for all $y \in \mathrm{Img}(\gamma)$, $b \in \mathrm{Img}(\gamma)$ and $g(b) = f(b)$, we conclude that $g(x) \le f(b)$. Therefore, by (2.6) we have

$$f(x_k) \le f(b) - \frac{f(b) - f(a)}{d(b,a)} d(x_k, b) \qquad \text{for all} \quad k.$$

Hence,

$$\limsup_{k\to\infty} f(x_k) \le f(b) - \frac{f(b) - f(a)}{d(b,a)} d(x, b).$$

Note that x is on the unique minimal geodesic connecting a and b, therefore $d(x, a) = d(a, b) - d(x, b)$, and

$$f(b) - \frac{f(b) - f(a)}{d(b,a)} d(x, b) = f(a) + \frac{f(b) - f(a)}{d(b,a)} d(x, a),$$

which implies

$$\limsup_{k\to\infty} f(x_k) \le f(a) + \frac{f(b) - f(a)}{d(b,a)} d(x, a).$$

Now consider the function $y \to g_{j_{n_k},k}(y) + \frac{d(y,x_k)}{k}$. This function attains a local minimum at x_k, therefore

$$0 \in \partial f(x_k) + \frac{f(b) - f(a)}{d(b,a)} \partial d_b(x_k) + j_{n_k} \partial d_{\mathrm{Img}(\gamma)}(x_k) + \frac{\partial d_x(x_k)}{k} + \frac{\partial d_{x_k}(x_k)}{k}.$$

Hence, there are (x_k^*), (u_k^*), (v_k^*), (w_k^*), (z_k^*) in $T_{x_k}M$ such that $x_k^* \in \partial f(x_k)$, $u_k^* \in \frac{f(b)-f(a)}{d(b,a)} \partial d_b(x_k)$, $v_k^* \in j_{n_k} \partial d_{\mathrm{Img}(\gamma)}(x_k)$, $w_k^* \in \frac{\partial d_x(x_k)}{k}$, $z_k^* \in \frac{\partial d_{x_k}(x_k)}{k}$ for all k and

$$x_k^* + u_k^* + v_k^* + w_k^* + z_k^* = 0 \qquad \text{for all } k.$$

From Lemma 2.2, $\langle v_k^*, \exp_{x_k}^{-1}(b)\rangle_{x_k} \le 0$ for all k. Note that $u_k^* = \frac{f(b)-f(a)}{d(b,a)d(x_k,b)}(-\exp_{x_k}^{-1}(b))$, therefore, $\langle u_k^*, \exp_{x_k}^{-1}(b)\rangle_{x_k} = -\frac{f(b)-f(a)}{d(b,a)} d(x_k, b)$. By Lemma 2.4,

$$\langle x_k^*, \exp_{x_k}^{-1}(b)\rangle_{x_k} \ge \frac{f(b) - f(a)}{d(b,a)} d(x_k, b) - 2\frac{d(x_k, b)}{k}.$$

Therefore,

$$\liminf_{k\to\infty} \langle x_k^*, \exp_{x_k}^{-1}(b)\rangle_{x_k} \ge \frac{f(b) - f(a)}{d(b,a)} d(b, x).$$

□

In the sequel, we always assume that M is a Hadamard manifold. Let $X(M)$ denote the set of all set-valued vector fields $A : M \rightrightarrows TM$ such that $A(x) \subset T_xM$ for each $x \in M$. The following definition extends the concepts of monotonicity for operators on Hilbert spaces to set-valued vector fields on Hadamard manifolds.

Definition 2.6. *Let $A : M \rightrightarrows TM$ be a set-valued vector field, it is said to be monotone iff for any $x, y \in \mathrm{dom}(A)$, we have*

$$\langle u, \exp_x^{-1}(y)\rangle_x \le \langle v, -\exp_y^{-1}(x)\rangle_y \qquad \textit{for every} \quad u \in A(x) \textit{ and } v \in A(y).$$

Example 2.7. *Let x be in a Hadamard manifold M. We consider $d_x : M \to \mathbb{R}$ defined by $d_x(y) := d(x,y)$. This function is locally Lipschitz and convex on M. Therefore, by Theorem 4.2 in* [11]*, the set-valued vector field $\partial d_x : M \rightrightarrows TM$ defined by*

$$\partial d_x(y) = \begin{cases} -\exp_y^{-1}(x)/d(x,y) & \text{if } x \neq y \\ B & \text{if } x = y \end{cases}$$

where B is the closed unit ball in T_yM, is monotone.

Now, we characterize convex lower semicontinuous functions on Hadamard manifolds in terms of the monotonicity of their subdifferential mappings. This result extends in some sense Theorem 2 of [17] and Theorem 4.2 in [11].

Theorem 2.8. *Let $f : M \to (-\infty, +\infty]$ be a proper lower semicontinuous function on a Hadamard manifold M. Then f is convex if and only if the set-valued mapping $\partial f : M \rightrightarrows TM$ is monotone.*

Proof. If f is convex, then $\partial f(x) = \partial^c f(x) = \partial_F f(x) = \partial_P f(x)$ for all $x \in \mathrm{dom}(f)$. Hence, it is enough to prove the monotonicity of $\partial^c f$. For any $x, y \in \mathrm{dom}(\partial^c f)$, there exists a unique geodesic $\gamma(t) = \exp_x(td)$ defined on $[0,1]$ connecting x and y, so there exists d such that $\exp_x(d) = y$. Moreover, there exists a unique geodesic $\sigma(t) = \exp_y(tg)$ defined on $[0,1]$ connecting y and x, which implies the existence of a vector $g \in T_yM$ such that $\exp_y(g) = x$. For any $u \in \partial^c f(x)$,

$$f(y) - f(x) \geq \langle u, \exp_x^{-1}(y)\rangle_x.$$

Moreover, for every $v \in \partial^c f(y)$,

$$f(x) - f(y) \geq \langle v, \exp_y^{-1}(x)\rangle_y.$$

By adding these two inequalities, we conclude that

$$\langle u, \exp_x^{-1}(y)\rangle_x \leq \langle v, -\exp_y^{-1}(x)\rangle_y.$$

For the converse, first we define the Moreau-Yosida proximal approximation corresponding to $\lambda > 0$ for the function f as

$$f_\lambda(x) := \inf_{y\in M}\{f(y) + \frac{1}{2\lambda}d(x,y)^2\}.$$

Then we prove the following statements:
(i) if ∂f is monotone, then $\partial f(x) = \partial^c f(x)$ for all $x \in M$.
(ii) if ∂f is monotone, then ∂f_λ is always nonempty.
(iii) if ∂f monotone, then ∂f_λ is monotone.
(iv) if $\mathrm{dom}(\partial f_\lambda) = M$ and ∂f_λ monotone, then f_λ is convex.
Eventually, since $f(x) = \sup_{\lambda>0} f_\lambda(x)$, f is also convex; see Theorem 11 in [5].

(i) If $\partial f(x) = \emptyset$, then the equality is true. Assume that $\partial f(x) \neq \emptyset$, therefore $f(x) \in \mathbb{R}$, and from the approximate mean value theorem for each d in T_xM such that $f(\exp_x(d)) \in \mathbb{R}$, there exist a sequence $v_k \to d$, a sequence $t_k \to t \in (0,1]$ and $x_k^* \in \partial f(\exp_x(t_k v_k))$ such that

$$f(\exp_x(d)) - f(x) \geq t^{-1} \limsup_{k\to\infty} \langle x_k^*, -\exp^{-1}_{\exp_x(t_k v_k)}(x)\rangle_{x_k}.$$

Since ∂f is monotone, we have for all $x^* \in \partial f(x)$,

$$\langle x^*, \exp_x^{-1}(\exp_x(t_k v_k))\rangle_x \leq \langle x_k^*, -\exp^{-1}_{\exp_x(t_k v_k)}(x)\rangle_{x_k}.$$

Hence,

$$f(\exp_x(d)) - f(x) \geq t^{-1} \limsup_{k\to\infty} \langle x^*, \exp_x^{-1}(\exp_x(t_k v_k))\rangle_x = \langle x^*, d\rangle_x.$$

This shows that $x^* \in \partial^c f(x)$, that is, $\partial f(x) \subset \partial^c f(x)$ which together with Lemma 1.6 proves our claim.

(ii) Assuming that ∂f is monotone, we conclude that $\partial f(x) = \partial^c f(x)$ for every $x \in M$. Therefore by Theorem 1.13, $\partial_P f$ is nonempty in a dense subset of dom$f$. Hence [5, Theorem 11] proves that f_λ is finite and locally Lipschitz. Then using Theorem 1.13 we have that ∂f_λ is always nonempty. Therefore $\text{dom}(\partial f_\lambda) = M$.

(iii) If ∂f monotone, we prove ∂f_λ is monotone. Indeed we first show that $\partial_P f_\lambda$ is monotone and then by Lemma 1.8, we get that ∂f_λ is monotone. Assume that $x^* \in \partial_P f_\lambda(x)$ and $y^* \in \partial_P f_\lambda(y)$, by [5, Theorem 11] f_λ is differentiable at x and y and $x^* = Df_\lambda(x)$, $y^* = Df_\lambda(y)$; see [5, Proposition 9]. Furthermore, $L_{x\bar{x}}(Df_\lambda(x)) \in \partial_P f(\bar{x})$, where $L_{x\bar{x}}$ is the parallel transport from x to $\bar{x}$ along the unique minimal geodesic connecting x and $\bar{x}$ and

$$f_\lambda(x) = f(\bar{x}) + \frac{1}{2\lambda} d(x,\bar{x})^2.$$

Moreover, $\frac{1}{\lambda}\exp_{\bar{x}}^{-1}(x) = L_{x\bar{x}}(Df_\lambda(x))$. Similar relations are also true if we replace x by y and $\bar{x}$ by $\bar{y}$. Note that

$$\begin{aligned}\langle x^*, -\exp_x^{-1}(y)\rangle_x + \langle y^*, -\exp_y^{-1}(x)\rangle_y \geq{}& \langle L_{x\bar{x}}(x^*), -\exp_{\bar{x}}^{-1}(\bar{y})\rangle_{\bar{x}} + \langle L_{y\bar{y}}(y^*), -\exp_{\bar{y}}^{-1}(\bar{x})\rangle_{\bar{y}}\\ &+ \langle x^*, -L_{\bar{x}x}(\exp_{\bar{x}}^{-1}(\bar{y})) - \exp_x^{-1}(y)\rangle_x\\ &+ \langle y^*, -L_{\bar{y}y}(\exp_{\bar{y}}^{-1}(\bar{x})) - \exp_y^{-1}(x)\rangle_y.\end{aligned}$$

Since ∂f is monotone and $\partial_P f(x) \subset \partial f(x)$ for every $x \in \text{dom}f$, we conclude that $\partial_P f$ is monotone, which implies

$$\langle L_{x\bar{x}}(x^*), -\exp_{\bar{x}}^{-1}(\bar{y})\rangle_{\bar{x}} + \langle L_{y\bar{y}}(y^*), -\exp_{\bar{y}}^{-1}(\bar{x})\rangle_{\bar{y}} \geq 0.$$

We also know that $d^2 : M \times M \to \mathbb{R}$ is smooth and convex. Moreover,

$$D(d^2)(x,\bar{x}) = (-2\exp_x^{-1}(\bar{x}), -2\exp_{\bar{x}}^{-1}(x)).$$

$$D(d^2)(y,\bar{y}) = (-2\exp_y^{-1}(\bar{y}), -2\exp_{\bar{y}}^{-1}(y)).$$

This shows that

$$\langle x^*, -L_{\bar{x}x}(\exp_{\bar{x}}^{-1}(\bar{y})) - \exp_x^{-1}(y)\rangle_x + \langle y^*, -L_{\bar{y}y}(\exp_{\bar{y}}^{-1}(\bar{x})) - \exp_y^{-1}(x)\rangle_y \geq 0.$$

Therefore $\partial_P f_\lambda$ is monotone and by Lemma 1.8 ∂f_λ is monotone.

(iv) Now, assume that x, y are two arbitrary points in M and γ defined by $\gamma(t) = \exp_x(t\exp_x^{-1}(y))$ is the unique geodesic connecting them, therefore for every $t_0 \in [0,1]$ and $u \in \partial f_\lambda(\gamma(t_0))$,

$$f_\lambda(x) - f_\lambda(\gamma(t_0)) \geq \langle u, \exp_{\gamma(t_0)}^{-1}(x)\rangle_{\gamma(t_0)}, \tag{2.8}$$

$$f_\lambda(y) - f_\lambda(\gamma(t_0)) \geq \langle u, \exp_{\gamma(t_0)}^{-1}(y)\rangle_{\gamma(t_0)}. \tag{2.9}$$

If we define $\sigma(s) = \gamma((1-s)t_0)$, then σ is a geodesic connecting $\gamma(t_0)$ and x. Therefore, $\sigma(s) = \exp_{\gamma(t_0)}(s\exp_{\gamma(t_0)}^{-1}(x))$ and $\exp_{\gamma(t_0)}^{-1}(x) = -t_0\gamma'(t_0)$. If we define $\eta(s) = \gamma((1-s)t_0 + s)$, then η is a geodesic connecting $\gamma(t_0)$ and y. Therefore,

$\eta(s) = \exp_{\gamma(t_0)}(s \exp^{-1}_{\gamma(t_0)}(y))$ and $\exp^{-1}_{\gamma(t_0)}(y) = (1-t_0)\gamma'(t_0)$. Hence, (2.8) and (2.9) imply that

$$(1-t_0)f_\lambda(x)+t_0 f_\lambda(y)-f_\lambda(\gamma(t_0)) \geq \langle u, -t_0(1-t_0)\gamma'(t_0)+t_0(1-t_0)\gamma'(t_0)\rangle_{\gamma(t_0)} = 0,$$

which shows that f_λ is convex. □

3. APPLICATION AND EXAMPLE

In this section, an example of a nonconvex function defined on the Poincaré half plane $\mathbb{H}$, which is a Hadamard manifold, is presented. By Theorem 2.8, we then conclude that the subdifferential set-valued map is not monotone. Moreover, a convex function on this Hadamard manifold is defined, then as a result of Theorem 2.8 its subdifferential map is monotone. These information can help us in choosing a suitable method to find minimizers of functions defined on Riemannian manifolds. Assume that we aim to find minimizers of a function h on $\mathbb{H}$. If ∂h is monotone, then we can apply Algorithm (4.3) in [22] to find $(u, v) \in \mathbb{H}$ such that $0 \in \partial h(u, v)$. Indeed, finding zeros of the set-valued map ∂h is equivalent to solving the following problem,

$$\min_{(u,v)\in\mathbb{H}} h(u, v). \tag{3.1}$$

Consider the Poincaré upper half plane $\mathbb{H} = \{(u, v) \in \mathbb{R}^2 : v > 0\}$ endowed with the Riemannian metric defined for every $(u, v) \in \mathbb{H}$ by

$$g_{ij}(u, v) = \frac{1}{v^2}\delta_{ij},$$

for $i, j = 1, 2$. The pair $(\mathbb{H}, g)$ is a Hadamard manifold with constant sectional curvature -1. It can be shown that the geodesics in $\mathbb{H}$ are the semi-lines and the semicircles orthogonal to the line $v = 0$; see [21, page 20].

The Riemannian distance between two points $(u_1, v_1), (u_2, v_2)$ in $\mathbb{H}$ is given by

$$d((u_1, v_1), (u_2, v_2)) = \operatorname{arccosh}\big(1 + \frac{(u_2-u_1)^2 + (v_2-v_1)^2}{2v_1v_2}\big).$$

We define $C := \{(u, v) \in \mathbb{H} : u^2 + v^2 \leq 3\}$ which is a closed and convex subset of $\mathbb{H}$. Let $f : \mathbb{H} \to (-\infty, +\infty]$ be defined by

$$f(u, v) = \begin{cases} |u| + v^4 - 2u^2 - 2v^2 + 3, & \text{if } (u, v) \in C; \\ +\infty & \text{if } (u, v) \notin C. \end{cases}$$

One can show that f is bounded from below and lower semicontinuous, but it is not a convex function on C. Now, we define a lower semicontinuous convex function h as follows

$$h(u, v) = \begin{cases} 1/v, & \text{if } (u, v) \in C; \\ +\infty & \text{if } (u, v) \notin C. \end{cases}$$

By Theorem 2.8, ∂f is not monotone, however ∂h is a monotone set-valued map. For finding minimizers of f on $\mathbb{H}$, we cannot use Algorithm (4.3) in [22], because ∂f is not monotone. But we have all the necessary conditions to use Algorithm (4.3) in [22] to find minimizers of h on $\mathbb{H}$. It worth mentioning that none of the two functions are locally Lipschitz on their domains.

References

[1] P. A. Absil, R. Mahony, R. Sepulchre, *Optimization Algorithm on Matrix Manifolds*, Princeton University Press, 2008.

[2] R. L. Adler, J. P. Dedieu, J. Y. Margulies, M. Martens, M. Shub, Newton's method on Riemannian manifolds and a geometric model for the human spine, IMA J. Numer. Anal., **22** (2002) 359-390.

[3] B. Afsari, R. Tron, R. Vidal, On the convergence of gradient descent for finding the Riemmanian center of mass, SIAM J. Control Optim., **51** (2013) 2230-2260.

[4] M. Alavi, S. Hosseini, M. R. Pouryayevali, On the calculus of limiting subjets on Riemannian manifolds, Mediterr. J. Math., **10** (2013) 593-607.

[5] D. Azagra, J. Ferrera, Proximal calculus on Riemannian manifolds, Mediterr. J. Math., **2** (2005) 437-450.

[6] D. Azagra, J. Ferrera, F. López-Mesas, Nonsmooth analysis and Hamilton-Jacobi equation on Riemannian manifolds, J. Funct. Anal., **220** (2005) 304-361.

[7] D. Azagra, J. Ferrera, F. López-Mesas, Approximate Rolle's theorems for the proximal subgradient and the generalized gradient, J. Math. Anal. Appl., **283** (2003) 180-191.

[8] D. Azagra, J. Ferrera, B. Sanz, Viscosity solutions to second order partial differential equations on Riemannian manifolds, J. Diffrential Equations., **245** (2008) 307-336.

[9] D. Azagra, J. Ferrera, Applications of proximal calculus to fixed point theory on Riemannian manifolds, Nonlinear. Anal., **67** (2007) 154-174.

[10] A. Barani, M. R. Pouryayevali, Invariant monotone vector fields on Riemannian manifolds, Nonlinear Anal., **70** (2009) 1850-1861.

[11] A. Barani, Generalized monotonicity and convexity for locally Lipschitz functions on Hadamard manifolds, Differ. Geom. Dyn. Syst., **15** (2013) 26-37.

[12] G. C. Bento, O. P. Ferreira, P. R. Oliveira, Local convergence of the proximal point method for a special class of nonconvex functions on Hadamard manifolds, Nonlinear Anal., **73** (2010) 564-572.

[13] G. C. Bento, O. P. Ferreira, P. R. Oliveira, Proximal point method for a special class of nonconvex functions on Hadamard manifolds, Optimization., **64**(2) (2015) 289-319.

[14] F. H. Clarke, Yu. S. Ledayaev, R. J. Stern, P. R. Wolenski, *Nonsmooth Analysis and Control Theory,* Graduate Texts in Mathematics, Vol. **178** Springer, New York, 1998.

[15] F. H. Clarke, Necessary conditions for nonsmooth problems in optimal control and the calculus of variations, Thesis, University of Washington, Seattle (1973).

[16] R. Correa, A. Jofre, L. Thibault, Characterization of lower semicontinuous convex functions, Proc. Amer. Math. Soc., **116** (1992) 67-72.

[17] O.P. Ferreira, Dini derivative and a characterization for Lipschitz and convex functions on Riemannian manifolds, Nonlinear Anal., **68** (2008),1517-1528.

[18] O. P. Ferreira, Proximal subgradient and a characterization of Lipschitz function on Riemannian manifolds, J. Math. Anal. Appl., **313** (2006) 587-597.

[19] N. Hadjisavvas, S. Komlosi, S. Schaible, *Handbook of Generalized Convexity and Generalized Monotonicity,* Springer, New York (2005).

[20] S. Hosseini, M. R. Pouryayevali, Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds, Nonlinear Anal., **74** (2011) 3884-3895.

[21] C. Udriste, *Convex Functions and Optimization Algorithms on Riemannian Manifolds,* Mathematics and Its Applications, **297**, Kluwer Academic, Dordrecht (1994).

[22] C. Li, G. López, V. Martín-Márquez, Monotone vector fields and the proximal point algorithm on Hadamard manifolds, J. Lond. Math. Soc., **79**(3) (2009) 663-683.

[23] C. Li, B. S. Mordukhovich, J. Wang, J. C. Yao, Weak sharp minima on Riemannian Manifolds, SIAM J. Optim., **21** (2011) 1523-1560.

[24] B. S. Mordukhovich, *Variational analysis and generalized differentiation. In: Basic Theory, vol. I. Applications,* **II**, Springer, Berlin (2006).

[25] S. Z. Nemeth, Monotone vector fields, Publ. Math. Debrecen., **54** (1999) 437-449.

[26] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, 1997.

[27] R. T. Rockafellar, Generalized directional derivatives and subgradients of nonconvex functions, Can. J. Math., **XXXII** (1980) 257-280.

[28] T. Sakai, *Riemannian Geometry,* Trans. Math. Monogor., Vol. **149**, Amer. Math. Soc., 1992.

[29] D. Zagrodny, Approximate mean value theorem for upper subderivatives, Nonlinear anal., **12** (1988) 1413-1428.