Mediterr. J. Math. 10 (2013), 593–607
DOI 10.1007/s00009-011-0171-3
1660-5446/13/010593-15, *published online* January 10, 2012

**Mediterranean Journal of Mathematics**

# On the Calculus of Limiting Subjets on Riemannian Manifolds

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**Abstract.** In this paper fuzzy calculus rules for subjets of order two on finite dimensional Riemannian manifolds are obtained. Then a second order singular subjet derived from a sequence of efficient subsets of symmetric matrices is introduced. Employing fuzzy calculus rules for subjets of order two and various qualification assumptions based on a second order singular subjet, calculus rules for limiting subjets on a finite dimensional Riemannian manifold are obtianed.

**Mathematics Subject Classification (2010).** Primary 49J52; Secondary 58C20.

**Keywords.** Subhessians; Subjets; Second order subdifferential calculus; Riemannian manifolds.

## 1. Introduction

The second order subjet of a function $f$ at some point $x$ as the Taylor expansion of a twice differentiable function which minorizes $f$ and coincides with $f$ at $x$ was introduced in [7]. A fuzzy calculus for subjets was first proved for real valued functions in [9]. Using the separable calculus rule which may be found in [8], the result of [9] was later shown to hold for extended real valued functions by Ioffe and Penot in [13]. Independently Eberhard and Nyblom proved an equivalent result via approximation by infimal regularization, without using the separable calculus rule; see [10]. In the paper by Ioffe and Penot [13], a qualification assumption was proposed that enabled the development of a subdifferential calculus for limiting subjets. It should be noted that the prominent role which first and second order generalized

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differentiability play in connection with many fields of mathematics such as optimization and control theory was widely recognized; see [6, 11].

The concept of generalized differentiability is often considered in finite dimensional or infinite dimensional Banach spaces where the linear structure of the space plays an important role. However, nondifferentiable, or nonsmooth functions arise naturally in many problems on smooth manifolds. A manifold is not necessarily a linear space therefore, new techniques are needed for adequately address nonsmooth problems on manifolds.

In the last few years several results have been obtained on various aspects of nonsmooth and variational analysis as well as their applications on Riemannian manifolds; see e. g., [1, 2, 5, 12, 14, 15]. In [3] the authors introduced second order subjets on Riemannian manifolds and carried out a systematic study of second order viscosity subdifferentials and viscosity solutions to second order partial differential equations on Riemannian manifolds.

Our aim is to obtain fuzzy calculus rules for subjets of order two on a finite dimensional Riemannian manifolds. Then using these fuzzy calculus rules and various qualification assumptions, calculus rules for limiting subjets are deduced; see [13]. We do not know whether the localization for the second order singular subjet through charts holds and it seems that our main results may not be proved by local techniques.

In this paper, we use the standard notations and known results of Riemannian manifolds. In what follows, $M$ is a finite dimensional manifold endowed with a Riemannian metric $\langle ., . \rangle_x$ on the tangent space $T_xM$. We identify (via the Riemannian metric) the tangent space of $M$ at a point $x$ with the cotangent space at $x$, denoted by $T_xM^*$. As usual, $\exp_x : U_x \to M$ will stand for the exponential function at $x$, where $U_x$ is an open subset of $T_xM$. Recall that the set $S$ in a Riemannian manifold $M$ is called convex if every two points $p_1,\ p_2 \in S$ can be joined by a unique geodesic whose image belongs to $S$.

The space of bilinear forms on $T_xM$ (respectively symmetric bilinear forms) is denoted by $\mathcal{L}^2(T_xM)$ (respectively $\mathcal{L}^2_s(T_xM)$). Elements of $\mathcal{L}^2_s(T_xM)$ will be denoted by letters $A,\ B,\ P,\ C,\ D$ and those of $T_xM^*$ by $\xi, \eta, x^*, y^*$. Also $L_s(T_xM)$ denotes the space of symmetric linear operators in $T_xM$ and $L(T_xM, T_yN)$ (respectively $L_s(T_xM, T_yN)$) denotes the space of linear operators (respectively symmetric linear operators ) from $T_xM$ to $T_yN$. For $A \in \mathcal{L}^2_s(T_xM),\ B \in \mathcal{L}^2_s(T_yN)$ and $C \in L(T_xM, T_yN)$ we set

$$Q(A, B, C) = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \in \mathcal{L}^2_s(T_xM \times T_yN),$$

where $*$ denotes the adjoint. Let $M$ be a Riemannian manifold and $x \in M$, the conic set $\mathcal{L}^{2+}_s(T_xM)$ of positive semidefinite symmetric bilinear forms induces a natural order on $\mathcal{L}^2_s(T_xM)$. For $A$ and $B$ in $\mathcal{L}^2_s(T_xM)$, we denote $A \geq B$ to mean $A - B \in \mathcal{L}^{2+}_s(T_xM)$.

By $i_M(x)$ we denote the injectivity radius of $M$ at $x$, that is the supremum of the radius $r$ of all balls $B(0_x, r)$ in $T_xM$ for which $\exp_x$ is a diffeomorphism from $B(0_x, r)$ onto $B(x, r)$. Similarly, $i(M)$ will denote the global injectivity radius of $M$, that is $i(M) = \inf\{i_M(x) : x \in M\}$.

For a minimizing geodesic $\gamma : [0, l] \to M$ connecting $x$ to $y$ in $M$, and for a vector $v \in T_xM$ there is a unique parallel vector field $P$ along $\gamma$ such that $P(0) = v$, this is called the parallel translation of $v$ along $\gamma$. The mapping $T_xM \ni v \mapsto P(l) \in T_yM$ is a linear isometry from $T_xM$ onto $T_yM$. This map is denoted by $L_{xy}$. Its inverse is of course $L_{yx}$. This isometry naturally induces an isometry (which we will still denote by $L_{xy}$), $T_xM^* \ni \xi \mapsto L_{xy}\xi \in T_yM^*$, defined by

$$\langle L_{xy}\xi, v\rangle_y := \langle \xi, L_{yx}v\rangle_x.$$

Similarly, $L_{xy}$ induces an isometry $\mathcal{L}^2(T_xM) \ni A \mapsto L_{xy}A \in \mathcal{L}^2(T_yM)$ defined by

$$\langle L_{xy}(A)v, v\rangle_y := \langle A(L_{yx}v), L_{yx}v\rangle_x. \tag{1.1}$$

Note that $L_{xy}$ is well defined when the minimizing geodesic which connects $x$ to $y$, is unique. For example, the parallel transport $L_{xy}$ is well defined when $x$ and $y$ are contained in a convex neighborhood. In what follows, $L_{xy}$ will be used wherever it is well defined.

It is worthwhile to mention that for Riemannian manifold $M \times N$,

$$L_{(x,y)(x_1,y_1)} : T_{(x,y)}(M \times N) \to T_{(x_1,y_1)}(M \times N)$$

is defined by

$$L_{(x,y)(x_1,y_1)}(v, w) := (L_{xx_1}v, L_{yy_1}w). \tag{1.2}$$

Recall that the Hessian $D^2\varphi$ of a $C^2$ smooth function $\varphi$ on a Riemannian manifold $M$ is defined by

$$D^2\varphi(X, Y) = \langle \nabla_X \nabla\varphi, Y\rangle,$$

where $\nabla\varphi$ is the gradient of $\varphi$, $X$, $Y$ are vector fields on $M$ and $\nabla_Y X$ denotes the covariant derivative of $X$ along $Y$(see [16, p. 31]). The Hessian is a symmetric tensor field of type $(0, 2)$ and, for a point $p \in M$, the value $D^2\varphi(X, Y)(p)$ only depends on $\varphi$ and the vectors $X(p)$, $Y(p) \in T_pM$. So we can define the second derivative of $\varphi$ at $p$ as the symmetric bilinear form $d^2\varphi(p) : T_pM \times T_pM \to \mathbb{R}$

$$(v, w) \mapsto d^2\varphi(p)(v, w) := D^2\varphi(X, Y)(p),$$

where $X$, $Y$ are any vector fields such that $X(p) = v$, $Y(p) = w$. A useful way to compute $d^2\varphi(p)(v, v)$ is to take geodesic $\gamma$ with $\gamma'(0) = v$ and calculate

$$\frac{d^2}{dt^2}\varphi(\gamma(t))\Big|_{t=0}.$$

We will often write $d^2\varphi(p)(v)^2$ instead of $d^2\varphi(p)(v, v)$.

Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function. The second order subjet of $f$ at a point $x \in \mathrm{dom} f = \{x \in M : \ f(x) < \infty\}$ is defined by

$J^{2,-}f(x) :=$

$$\{(d\varphi(x), d^2\varphi(x)) : \varphi \in C^2(M, \mathbb{R}), f - \varphi \text{ attains a local minimum at } x\}.$$

If $(\xi, A) \in J^{2,-}f(x)$, we say that $\xi$ is a first order subdifferential of $f$, and $A$ is a second order subdifferential of $f$ at $x$. Note that $J^{2,-}f(x)$ is a subset of $T_xM^* \times \mathcal{L}_s^2(T_xM)$.

Recall that a sequence $(A_n)$ with $A_n \in \mathcal{L}_s^2(T_{x_n}M)$ is said to converge to $A \in \mathcal{L}_s^2(T_xM)$ provided $x_n$ converges to $x$ in $M$ and for every vector field $V$ defined on an open neighborhood of $x$ we have that $\langle A_nV(x_n), V(x_n)\rangle$ converges to $\langle AV(x), V(x)\rangle$.

Similarly, a sequence $(\xi_n)$ with $\xi_n \in T_{x_n}M^*$ converge to $\xi$ provided that $x_n \to x$ and $\langle \xi_n, V(x_n)\rangle \to \langle \xi, V(x)\rangle$.

For a lower semicontinuous function $f : M \to (-\infty, +\infty]$ defined on a Riemannian manifold $M$, the second order limiting subjet of $f$ at a point $x \in M$ is defined by (see [3])

$$\begin{aligned}\bar{J}^{2,-}f(x) :=& \{(\xi, A) \in T_xM^* \times \mathcal{L}_s^2(T_xM) : \exists x_n \in M,\ \exists(\xi_n, A_n) \in J^{2,-}f(x_n) \\ & \text{s.t. } (x_n, f(x_n), \xi_n, A_n) \to (x, f(x), \xi, A)\}.\end{aligned} \tag{1.3}$$

The following proposition is well known for the case when $M = \mathbb{R}^n$. This known result can be extended to the Riemannian setting (see [3, p. 313]).

**Proposition 1.1.** *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function. Suppose that $x \in \operatorname{dom}(f)$, $\xi \in T_xM^*$ and $A \in \mathcal{L}_s^2(T_xM)$. Then the following statements are equivalent:*

(a) $(\xi, A) \in J^{2,-}f(x)$.
(b) $f(\exp_x(v)) \geq f(x) + \langle \xi, v\rangle_x + \frac{1}{2}\langle Av, v\rangle_x + o(\|v\|^2)$.
(c) $\liminf_{\|v\|\to 0} \|v\|^{-2}(f \circ \exp_x(v) - f \circ \exp_x(0) - \langle \xi, v\rangle_x - \frac{1}{2}\langle Av, v\rangle_x) \geq 0$.
(d) *For any $\varepsilon > 0$ the function*

$$v \mapsto f \circ \exp_x(v) - f \circ \exp_x(0) - \langle \xi, v\rangle_x - \frac{1}{2}\langle Av, v\rangle_x + \varepsilon\|v\|^2$$

*has a local minimum at $0_x$.*

## 2. Fuzzy calculus rules for subjets of order two

In order to prove fuzzy calculus rules for second order subjets on Riemannian manifolds, the following lemmas are needed.

**Lemma 2.1.** ([1, Lemma 6.5]). *Let $M$ be a Riemannian manifold and $x_0, y_0 \in M$ be such that $d(x_0, y_0) < \min\{i_M(x_0), i_M(y_0)\}$. Then*

$$L_{y_0x_0}\left(\frac{\partial d(x_0, y_0)}{\partial y}\right) = -\frac{\partial d(x_0, y_0)}{\partial x}.$$

**Lemma 2.2.** ([3, Proposition 3.1]). *Let $M$ be a Riemannian manifold. Consider the function $\varphi(x, y) = d(x, y)^2$ defined on $M \times M$. Assume that $M$ has positive sectional curvature. Then*

$$d^2\varphi(x, y)(v, L_{xy}v)^2 \leq 0,$$

*for all* $v \in T_xM$ *and* $x, y \in M$ *with* $d(x,y) < \min\{i_M(x), i_M(y)\}$.

**Lemma 2.3.** ([3, Theorem 2.10]). *Let* $M_1, \dots, M_k$ *be Riemannian manifolds, and* $\Omega_i \subset M_i$ *be open subsets. Define* $\Omega = \Omega_1 \times \cdots \times \Omega_k \subset M_1 \times \cdots \times M_k = M$. *Let* $u_i$ *be lower semicontinuous functions on* $\Omega_i$, $i = 1, \dots, k$, *let* $\varphi$ *be a* $C^2$ *smooth function on* $\Omega$ *and set*

$$\omega(x) = u_1(x_1) + \cdots + u_k(x_k)$$

*for* $x = (x_1, \dots, x_k) \in \Omega$. *Assume that* $\hat{x} = (\hat{x}_1, \dots, \hat{x}_k)$ *is a local minimum of* $\omega - \varphi$. *Then, for each* $\varepsilon > 0$ *there exist bilinear forms* $A_i \in \mathcal{L}_s^2(T_{\hat{x}_i}M_i), i = 1, \dots, k$, *such that*

$$\left(\frac{\partial}{\partial x_i}\varphi(\hat{x}), A_i\right) \in \bar{J}^{2,-}u_i(\hat{x}_i)$$

*for* $i = 1, \dots, k$, *and the block diagonal matrix with entries* $A_i$ *satisfies*

$$\begin{pmatrix} A_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_k \end{pmatrix} \geq H - \varepsilon I,$$

*where* $H = d^2\varphi(\hat{x}) \in \mathcal{L}_s^2(T_{\hat{x}}M)$.

*Remark* 2.4. Let $M$ be a Riemannian manifold. Then:
(a) An easy consequence of the definition of the parallel translation along a curve as a solution to an ordinary linear differential equation, implies that the mapping

$$C : TM^* \to T_{x_0}M^*, \; C(x, \xi) = L_{xx_0}(\xi),$$

is continuous at $(x_0, \xi_0)$, that is, if $(x_n, \xi_n) \to (x_0, \xi_0)$ in $TM^*$ then $L_{x_nx_0}(\xi_n) \to L_{x_0x_0}(\xi_0) = \xi_0$, for every $(x_0, \xi_0) \in TM^*$; see [1, Remark 6.11].
(b) By the continuity properties of the parallel transport and the geodesic, see [4, Theorem 35], for fixed point $z \in M$ and for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that:

$$\|L_{xy}L_{zx} - L_{zy}\| \leq \varepsilon \;\text{ provided that } d(x,y) < \delta.$$

**Theorem 2.5.** *Let* $M$ *be a complete Riemannian manifold with positive sectional curvature. Suppose that* $f, g$ *are functions on* $M$ *which are lower semicontinuous near* $x$ *and finite at* $x$ *and* $(\xi, A) \in J^{2,-}(f+g)(x)$. *Then for any* $\varepsilon > 0$ *there are* 2 *triples* $(x_i, \xi_i, A_i), i = 1, 2$, *such that,* $(\xi_1, A_1) \in J^{2,-}f(x_1)$, $(\xi_2, A_2) \in J^{2,-}g(x_2)$, $d(x_i, x) < \varepsilon$, *for* $i = 1, 2$, $|f(x_1) - f(x)| < \varepsilon$, $|g(x_2) - g(x)| < \varepsilon$,

$$\|L_{x_1x}(\xi_1) + L_{x_2x}(\xi_2) - \xi\| < \varepsilon,$$

*and*

$$L_{x_1x}(A_1) + L_{x_2x}(A_2) \geq A - \varepsilon I.$$

*Proof.* Without loss of generality we may assume that $\xi = 0$, $A = 0$. Since both $f$ and $g$ are lower semicontinuous, it follows that for given $\varepsilon > 0$, there exists $\rho > 0$ such that

$$f(y) \geq f(x) - \varepsilon, \quad g(y) \geq g(x) - \varepsilon \qquad \text{if} \quad d(y,x) < \rho. \tag{2.1}$$

Proposition 1.1 implies the existence of a real number $\delta$ with $0 < \delta < \min(\varepsilon, \rho)$ such that

$$(f+g)\circ \exp_x(v) - (f+g)\circ \exp_x(0) + \frac{1}{2}\varepsilon\|v\|_x^2 > 0 \quad \text{if} \quad \|v\|_x \le \delta,\ v \neq 0.$$

Therefore

$$f(z) + g(z) - f(x) - g(x) + \frac{1}{2}\varepsilon d(z,x)^2 > 0 \quad \text{if} \quad d(z,x) \le \delta,\ z \neq x. \tag{2.2}$$

Now we define $h_n$ on $M \times M$ as follows:

$$h_n(p,q) := f\circ \exp_p(0) + g\circ \exp_q(0) + \frac{n}{2}d(p,q)^2 + \frac{\varepsilon}{2}(d(p,x)^2 + d(x,q)^2) - f\circ \exp_x(0) - g\circ \exp_x(0).$$

We have that $h_n(x,x) = 0$, and $h_n(p,q) \ge -2\varepsilon$ if $d(x,q) \le \delta$, $d(p,x) \le \delta$. By the completeness of $M$ it follows that $h_n$ attains its minimum on $\bar{B}(x,\delta) \times \bar{B}(x,\delta)$ at a point $(p_n, q_n)$.

As $h_n(p_n,q_n) \le h_n(x,x) = 0$, we conclude from (2.1) that

$$\frac{n}{2}d(p_n,q_n)^2 \le 2\varepsilon,$$

which means that $d(p_n,q_n) \to 0$ as $n \to \infty$.

Let us extract a subsequence of $(p_n,q_n)$ (without relabelling) such that $(p_n,q_n) \to (w,w)$. Then by lower semicontinuity of $h_n(p,q)$

$$f(w) + g(w) - f(x) - g(x) + \varepsilon d(w,x)^2 \le \liminf_{n\to\infty} h_n(p_n,q_n) \le 0,$$

and (2.2) implies that $w = x$. Thus $(p_n) \to x$ and $(q_n) \to x$, in particular $d(p_n,x) < \min(\frac{\varepsilon}{2},\frac{1}{2})$, $d(x,q_n) < \min(\frac{\varepsilon}{2},\frac{1}{2})$, $|f(p_n) - f(x)| < \frac{\varepsilon}{2}$ and $|g(q_n) - g(x)| < \frac{\varepsilon}{2}$ for large $n$. Moreover, Remark 2.4 implies that for sufficiently large $n$,

$$\|L_{p_n x}L_{q_n p_n} - L_{q_n x}\| < \varepsilon.$$

On the other hand, for $n$ large enough $d(p_n,q_n) < i(\bar{B}(x,\delta))$. Now we fix such $n$ and consider $p_n$, $q_n$. Then we apply Lemma 2.3 with $u_1 = f, u_2 = g$ and

$$\varphi(p,q) = -\frac{1}{2}(nd(p,q)^2 + \varepsilon d(p,x)^2 + \varepsilon d(x,q)^2) + f(x) + g(x). \tag{2.3}$$

Hence there exist bilinear forms $B_{1,n} \in \mathcal{L}_s^2(T_{p_n}M), B_{2,n} \in \mathcal{L}_s^2(T_{q_n}M)$ such that

$$(\frac{\partial}{\partial p}\varphi(p_n,q_n), B_{1,n}) \in \bar{J}^{2,-}f(p_n) \quad \text{and} \quad (\frac{\partial}{\partial q}\varphi(p_n,q_n), B_{2,n}) \in \bar{J}^{2,-}g(q_n).$$

Set

$$\eta_{1,n} := \frac{\partial}{\partial p}\varphi(p_n,q_n) = -\frac{1}{2}\left(2nd(p_n,q_n)\frac{\partial d}{\partial p}(p_n,q_n) + 2\varepsilon d(p_n,x)\frac{\partial d}{\partial p}(p_n,x)\right) = n\exp_{p_n}^{-1}(q_n) + \varepsilon \exp_{p_n}^{-1}(x). \tag{2.4}$$

The third equality can be checked, for instance, by using the first formula of the arc-length (see [16, p. 90]). Also set

$$\eta_{2,n} := \frac{\partial}{\partial q}\varphi(p_n, q_n) = -\frac{1}{2}\left(2nd(p_n, q_n)\frac{\partial d}{\partial q}(p_n, q_n) + 2\varepsilon d(x, q_n)\frac{\partial d}{\partial q}(x, q_n)\right)$$
$$= n\exp_{q_n}^{-1}(p_n) + \varepsilon \exp_{q_n}^{-1}(x). \tag{2.5}$$

Without loss of generality we can suppose that

$$k = \max\{\|B_{1,n}\|, \|B_{2,n}\|, \|\eta_{1,n}\|, \|\eta_{2,n}\|\} \leq 1.$$

Otherwise we change $\varphi$ into

$$\varphi(p, q) = -\frac{1}{2k}(nd(p, q)^2 + \varepsilon d(p, x)^2 + \varepsilon d(x, q)^2) + f(x) + g(x),$$

and $B_{i,n}$, $i = 1, 2$ into $\frac{B_{i,n}}{k}$.

On the other hand,

$$\begin{aligned}\eta_{1,n} + L_{q_n p_n}(\eta_{2,n}) =& n\exp_{p_n}^{-1}(q_n) + \varepsilon\exp_{p_n}^{-1}(x)\\ &+ L_{q_n p_n}\left(n\exp_{q_n}^{-1}(p_n) + \varepsilon\exp_{q_n}^{-1}(x)\right)\\ =&\varepsilon\exp_{p_n}^{-1}(x) + \varepsilon L_{q_n p_n}(\exp_{q_n}^{-1}(x)).\end{aligned}$$

Note that Lemma 2.1 implies that $L_{q_n p_n}(n\exp_{q_n}^{-1}(p_n)) = -n\exp_{p_n}^{-1}(q_n)$, so the last equality is true.

Since parallel translation preserve the norm, we have

$$\begin{aligned}\|L_{p_n x}\left(\eta_{1,n} + L_{q_n p_n}(\eta_{2,n})\right)\| &= \|\eta_{1,n} + L_{q_n p_n}(\eta_{2,n})\|\\ &= \|\varepsilon\exp_{p_n}^{-1}(x) + \varepsilon L_{q_n p_n}(\exp_{q_n}^{-1}(x))\|\\ &\leq \varepsilon\|\exp_{p_n}^{-1}(x)\| + \varepsilon\|\exp_{q_n}^{-1}(x)\|\\ &= \varepsilon d(p_n, x) + \varepsilon d(x, q_n) < \varepsilon\frac{1}{2} + \varepsilon\frac{1}{2} = \varepsilon.\end{aligned} \tag{2.6}$$

Using the smoothness of the function $(x_1, x_2) \mapsto d(x_1, x_2)^2$, we deduce that the function $F(p, q) := d(p, x)^2 + d(x, q)^2$ is smooth. Hence, there exists $k_1 > 0$ such that

$$\frac{1}{2}d^2F(p_n, q_n)(v, L_{p_n q_n}(v))^2 \leq k_1, \quad \text{for all } v \in T_{p_n}M,\ ||v|| = 1.$$

Lemma 2.2 implies $-nd^2(d^2)(p_n, q_n)(v, L_{p_n q_n}(v))^2 \geq 0$. Therefore,

$$d^2\varphi(p_n, q_n)(v, L_{p_n q_n}(v))^2 \geq -k_1\varepsilon \ \text{ for all } \ v \in T_{p_n}M,\ \|v\| = 1.$$

Without loss of generality we can assume that

$$d^2\varphi(p_n, q_n)(v, L_{p_n q_n}(v))^2 \geq -\varepsilon \ \text{ for all } \ v \in T_{p_n}M,\ \|v\| = 1.$$

Therefore Lemma 2.3 implies that

$$\langle B_{1,n}v, v\rangle + \langle B_{2,n}L_{p_n q_n}(v), L_{p_n q_n}(v)\rangle \geq -\varepsilon - \varepsilon I(v, L_{p_n q_n}(v))^2.$$

Hence

$$\langle B_{1,n}v, v\rangle + \langle B_{2,n}L_{p_n q_n}(v), L_{p_n q_n}(v)\rangle \geq -\varepsilon - \varepsilon(\|v\|^2 + \|L_{p_n q_n}(v)\|^2).$$

Now assume that $v = L_{xp_n}(u)$, where $u$ is an arbitrary unite vector in $T_xM$. Thus

$$\langle B_{1,n}L_{xp_n}(u), L_{xp_n}(u)\rangle + \langle B_{2,n}L_{p_nq_n}(L_{xp_n}(u)), L_{p_nq_n}(L_{xp_n}(u))\rangle \geq -\varepsilon - 2\varepsilon\|L_{xp_n}(u)\|^2.$$

By Definition 1.1

$$\langle L_{p_nx}(B_{1,n})u, u\rangle + \langle L_{p_nx}(L_{q_np_n}B_{2,n})u, u\rangle \geq -\varepsilon - 2\varepsilon\|u\|^2 = -3\varepsilon. \tag{2.7}$$

Hence, we proved that there exist $(\eta_{1,n}, B_{1,n}) \in \bar{J}^{2,-}f(p_n)$, $(\eta_{2,n}, B_{2,n}) \in \bar{J}^{2,-}g(q_n)$ such that $d(p_n, x) < \varepsilon/2,\ d(x, q_n) < \varepsilon/2$ and $|f(p_n) - f(x)| < \varepsilon/2$, $|g(q_n) - g(x)| < \varepsilon/2$.

By (1.3) there exists $(\eta_{1,m}, B_{1,m}) \in J^{2,-}f(p_{n,m})$ such that $p_{n,m} \to p_n$, $f(p_{n,m}) \to f(p_n),\ \eta_{1,m} \to \eta_{1,n}, and B_{1,m} \to B_{1,n}$.

Also, there exists $(\eta_{2,m}, B_{2,m}) \in J^{2,-}g(q_{n,m})$ such that $q_{n,m} \to q_n$, $g(q_{n,m}) \to g(q_n),\ \eta_{2,m} \to \eta_{2,n}$, and $B_{2,m} \to B_{2,n}$. Now, for $m$ large enough

$$\begin{aligned}\|L_{p_{n,m}x}(\eta_{1,m}) + L_{q_{n,m}x}(\eta_{2,m})\| &\leq \|L_{p_{n,m}x}(\eta_{1,m}) - L_{p_nx}(\eta_{1,n})\| \\ &\quad + \|L_{p_nx}(\eta_{1,n}) + L_{p_nx}L_{q_np_n}(\eta_{2,n})\| \\ &\quad + \|L_{q_nx}(\eta_{2,n}) - L_{q_{n,m}x}(\eta_{2,m})\| \\ &\quad + \|L_{p_nx}L_{q_np_n}(\eta_{2,n}) - L_{q_nx}(\eta_{2,n})\| \\ &\leq 4\varepsilon.\end{aligned}$$

Similarly, one can deduce that

$$L_{p_{n,m}x}(B_{1,m}) + L_{q_{n,m}x}(B_{2,m}) \geq -6\varepsilon I.$$

For large $m$, $d(p_{n,m}, p_n) < \varepsilon/2,\ d(q_{n,m}, q_n) < \varepsilon/2,\ |f(p_{n,m}) - f(p_n)| < \varepsilon/2$, and $|g(q_{n,m}) - g(q_n)| < \varepsilon/2$. Now we fix such an $m$ and call it $\alpha$. It remains to set $x_1 = p_{n,\alpha},\ x_2 = q_{n,\alpha},\ \xi_1 = \eta_{1,\alpha},\ \xi_2 = \eta_{2,\alpha},\ A_1 = B_{1,\alpha},\ A_2 = B_{2,\alpha}$. □

The following proposition is concerned with the composition operation. It is proved in the case when $M$ and $N$ are finite dimensional Euclidean spaces, see [13]. In a similar way one can prove the case when $M$ and $N$ are finite dimensional Riemannian manifolds.

**Proposition 2.6.** *Let $M$ and $N$ be Riemannian manifolds, $x \in M$, $y \in N$. Suppose that $g : N \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous near $y$ and finite at $y$. Consider the function $f$ on $M \times N$ given by $f(p, q) = g(q)$.*

(a) *For any $(\eta, B) \in J^{2,-}g(y)$ and $A \in \mathcal{L}_s^2(T_xM)$, $A \leq 0$, $S \in L(T_xM, T_yN)$, one has, setting $D := B + S \circ A \circ S^*$ and $C := S \circ A$,*

$$((0, \eta), Q(A, D, C)) \in J^{2,-}f(x, y),$$

*in particular, $((0, \eta), Q(0, B, 0)) \in J^{2,-}f(x, y)$.*

(b) *Conversely, if $((\xi, \eta), P) \in J^{2,-}f(x, y)$* and *$P = Q(A, D, C)$,* then *$\xi = 0,\ A \leq 0$ and for any positive $\alpha$ there exist an $S \in L(T_xM, T_yN)$, such that $C = S \circ (A - \alpha I)$, $B := D - S \circ (A - \alpha I) \circ S^*$, $(\eta, B) \in J^{2,-}g(y)$ and $Q(0, B, 0) \geq P - Q(\alpha I, 0, 0)$.*

We proceed now to derive fuzzy chain rule for second order subjets on Riemannian manifolds.

**Theorem 2.7.** *Let $M$ be a complete Riemannian manifold with positive sectional curvature and $F : M \to \mathbb{R}^n$ be a map of class $C^1$ near $x \in M$. Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous near $y = F(x)$ and finite at $y$. Set $f = g \circ F$ and assume that $(\xi^*, A) \in J^{2,-}f(x)$. Then, for each $\varepsilon > 0$, there are $z \in M$, $\xi \in T_zM^*$, $r$, $\zeta$, $\eta \in \mathbb{R}^n$ and $B \in \mathcal{L}^2_s(\mathbb{R}^n)$, $C \in \mathcal{L}^2_s(T_zM)$ such that $d(z, x) < \varepsilon$, $\|r - y\| < \varepsilon$, $\|L_{zx}\xi - \xi^*\| < \varepsilon$, $\|\zeta - \eta\| < \varepsilon$ and*

$$(\xi, C) \in J^{2,-}(\eta \circ F)(z), \ (\zeta, B) \in J^{2,-}g(r), \tag{2.8}$$

$$C + dF(z)^* \circ B \circ dF(z) \geq L_{xz}A - \varepsilon I. \tag{2.9}$$

*Proof.* Let $x \in M$, $y = F(x)$. Since $F$ is $C^1$, there exists $k > 0$ such that for $\varepsilon > 0$ small enough and for all $q \in M$ with $d(q, x) \leq \varepsilon$, $\|dF(q)\| \leq k$.

Let $h_1(p, r) = g(r)$ and $h_2$ denote the indicator function of graph $F$. Moreover, assume that $h = h_1 + h_2$. Obviously, for each $(p, r)$, $h(p, r) \geq f(p)$. Therefore for every $\delta > 0$,

$$h(p, r) - h(x, y) \geq f \circ \exp_x(u) - f \circ \exp_x(0) \geq \langle \xi^*, u\rangle + \frac{1}{2}[\langle Au, u\rangle - \delta\|u\|^2],$$

for all $u \in T_xM$ provided that $\|u\|_x$ is sufficiently small. By Proposition 1.1

$$((\xi^*, 0), Q(A, 0, 0)) \in J^{2,-}h(x, y).$$

Hence Theorem 2.5 implies the existence of $x_i, y_i, x_i^*, y_i^*$ and $P_i, i = 1, 2,$ such that

$$\begin{aligned} &((x_i^*, y_i^*), P_i) \in J^{2,-}h_i(x_i, y_i), \qquad i = 1, 2, \\ &L_{(x_1,y_1)(x,y)}P_1 + L_{(x_2,y_2,)(x,y)}P_2 \geq Q(A, 0, 0) - \varepsilon(2 + 2k^2)^{-1}I, \\ &\|L_{x_1x}(x_1^*) + L_{x_2x}(x_2^*) - \xi^*\|_x < \varepsilon, \quad \|y_1^* + y_2^*\| < \varepsilon, \\ &d(x_i, x) < \varepsilon, \quad \|y_i - y\| < \varepsilon, \quad i = 1, 2. \end{aligned} \tag{2.10}$$

By Proposition 2.6 we have $x_1^* = 0$, and there is a $B \in \mathcal{L}^2_s(\mathbb{R}^n)$ such that $(y_1^*, B) \in J^{2,-}g(y_1)$ and $Q(0, B, 0) \geq P_1 - Q(\varepsilon(2+2k^2)^{-1}I, 0, 0)$, so that

$$L_{(x_1,y_1)(x,y)}Q(0, B, 0) + L_{(x_2,y_2)(x,y)}P_2 \geq Q(A, 0, 0) - \varepsilon(1 + k^2)^{-1}I.$$

Hence

$$\begin{aligned} &L_{(x,y)(x_2,y_2)}L_{(x_1,y_1)(x,y)}Q(0, B, 0) + P_2 \\ &\qquad \geq L_{(x,y)(x_2,y_2)}Q(A, 0, 0) - L_{(x,y)(x_2,y_2)}\varepsilon(1 + k^2)^{-1}I. \end{aligned} \tag{2.11}$$

We define $G : U(0_{x_2}) \subset T_{x_2}M \to \mathbb{R}^n$ by

$$G(u) := F \circ \exp_{x_2}(u) - F \circ \exp_{x_2}(0).$$

Then the inclusion for $i = 2$ in (2.10) implies that for each $\delta > 0$,

$$0 \geq \langle x_2^*, u\rangle + \langle y_2^*, G(u)\rangle + \frac{1}{2}\langle P_2(u, G(u)), (u, G(u))\rangle - \delta(\|u\|^2 + \|G(u)\|^2), \tag{2.12}$$

for all sufficiently small $u \in U(0_{x_2})$.

Note that since $F$ is differentiable at $x_2$ it follows that

$$G(u) = dF(x_2)(u) + o(\|u\|). \tag{2.13}$$

Furthermore, we define $C \in \mathcal{L}_s^2(T_{x_2}M)$ by

$$\langle Cu, u\rangle := \langle P_2(u, dF(x_2)(u)), (u, dF(x_2)(u))\rangle.$$

Now, considering (2.11) at $(u, dF(x_2)(u))$, using (1.1) and (1.2) we conclude that

$$\langle dF(x_2)^* \circ B \circ dF(x_2)u, u\rangle + \langle Cu, u\rangle \geq \langle A(L_{x_2x}u), L_{x_2x}u\rangle - \langle \varepsilon Iu, u\rangle$$

which means

$$dF(x_2)^* \circ B \circ dF(x_2) + C \geq L_{xx_2}A - \varepsilon I.$$

On the other hand, we obtain from (2.12) and (2.13) that

$$-\langle y_2^*, F \circ \exp_{x_2}(u) - F \circ \exp_{x_2}(0)\rangle \geq \langle x_2^*, u\rangle + \frac{1}{2}\langle Cu, u\rangle - \delta(1 + k^2)\|u\|^2,$$

provided $u$ is sufficiently close to zero in $T_{x_2}M$. As $\delta$ is an arbitrary positive number we get

$$(x_2^*, C) \in J^{2,-}(-y_2^* \circ F)(x_2),$$

and we arrive at the proof if we set $r = y_1, \zeta = y_1^*,\ \eta = -y_2^*,\ z = x_2,\ \xi = x_2^*$. □

## 3. Main Results

In this section using fuzzy calculus rules obtained in the previous section, calculus rules for limiting subjets are deduced.

**Definition 3.1.** Let $M$ be a Riemannian manifold and $x \in M$. Suppose that $\mathcal{A}$ is a subset of $\mathcal{L}_s^2(T_xM)$ and $A \in \mathcal{A}$, we say $A$ is efficient in $\mathcal{A}$ if $\|A\| = \min\{\|B\| :\ B \geq A,\ B \in \mathcal{A}\}$.

Suppose that $f : M \to (-\infty, +\infty]$ is a lower semicontinuous function defined on a Riemannian manifold $M$, $x \in \mathrm{dom}(f)$. Consider the set

$$J^{2,-}f(x, \xi) := \{A \in \mathcal{L}_s^2(T_xM) :\ (\xi, A) \in J^{2,-}f(x)\ \}.$$

The efficient elements of $J^{2,-}f(x, \xi)$ denoted by $J^{2,e}f(x, \xi)$ will be called efficient subhessians of $f$ at $x$ for $\xi$. Moreover, we define the second order efficient subjet of $f$ at $x$ denoted by $J^{2,e}f(x)$ as

$$J^{2,e}f(x) := \{(\xi, A) :\ A \in J^{2,e}f(x, \xi)\}.$$

Since $J^{2,-}f(x, \xi)$ is a closed subset of $\mathcal{L}_s^2(T_xM)$, it has an element of least norm.

The following lemma is a direct consequence of [3, corollary 2.3].

**Lemma 3.2.** *Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function defined on a Riemannian manifold $M$, $x \in \mathrm{dom}(f)$. Then*

$$(\xi, A) \in J^{2,e}f(x) \Leftrightarrow (\xi, A) \in J^{2,e}(f \circ \exp_x)(0_x).$$

**Definition 3.3.** Let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function defined on a Riemannian manifold $M$, $x \in \mathrm{dom}(f)$. We define the second order singular subjet of $f$ at $x$ denoted by $J^{2,\infty} f(x)$ as

$$\begin{aligned} J^{2,\infty} f(x) :=& \{(\xi, A) : \exists x_n \in M,\ \exists (\xi_n, A_n) \in J^{2,e} f(x_n),\ \exists \lambda_n \in (0, +\infty) \\ & \text{s.t. } (\lambda_n, x_n, f(x_n), \lambda_n \xi_n, \lambda_n A_n) \to (0, x, f(x), \xi, A)\}. \end{aligned} \tag{3.1}$$

**Definition 3.4.** Let $M$ be a Riemannian manifold, and let $F : M \to \mathbb{R}^n$ be continuous at $x$, we define the second order singular cosubjet of $F$ at $x$ denoted by $D^2_\infty F(x, y)$ as follows,

$$\begin{aligned} D^2_\infty F(x, y) :=& \{(\xi, A) : \exists x_n \in M,\ \exists z_n \in \mathbb{R}^n,\ \exists (\xi_n, A_n) \in J^{2,e} \langle z_n, F(x_n)\rangle, \\ & \exists \lambda_n > 0 \text{ s.t. } (\lambda_n, z_n, x_n, \lambda_n \xi_n, \lambda_n A_n) \to (0, y, x, \xi, A)\}. \end{aligned} \tag{3.2}$$

**Definition 3.5.** Let $M$ be a Riemannian manifold, and let $F : M \to \mathbb{R}^n$ be continuous at $x$, we define the second order singular cojet of $F$ at $x$ denoted by $\overline{D}^2 F(x, y)$ as follows,

$$\begin{aligned} \overline{D}^2 F(x, y) :=& \{(\xi, A) : \exists x_n \in M,\ \exists z_n \in \mathbb{R}^n,\ \exists (\xi_n, A_n) \in J^{2,-} \langle z_n, F(x_n)\rangle, \\ & \text{s.t. } (z_n, x_n, \xi_n, A_n) \to (y, x, \xi, A)\}. \end{aligned} \tag{3.3}$$

We do not know whether the analogue of Lemma 3.2 for the second order singular subjet holds.

**Theorem 3.6.** *Let $M$ be a complete Riemannian manifold with positive sectional curvature and $f,\ g : M \to (-\infty, +\infty]$ be lower semicontinuous functions. Suppose that $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and the following assumption holds:*

*if $(\xi_1, A_1) \in J^{2,\infty} f(x)$, $(\xi_2, A_2) \in J^{2,\infty} g(x)$ and $\xi_1 + \xi_2 = 0$, $A_1 + A_2 \geq 0$, then*

$\xi_1 = \xi_2 = 0$ *and* $A_1 \geq 0,\ A_2 \geq 0$.

*Then*

$$\bar{J}^{2,-}(f + g)(x) \subset \bar{J}^{2,-} f(x) + \bar{J}^{2,-} g(x).$$

*Proof.* Assume that $(\xi, A) \in \bar{J}^{2,-}(f + g)(x)$. Hence there exist $x_n \in M$, $(\xi_n, A_n) \in J^{2,-}(f + g)(x_n)$ such that $(x_n, (f + g)(x_n), \xi_n, A_n) \to (x, (f + g)(x), \xi, A)$.

Theorem 2.5 implies that for a given positive sequence $(\varepsilon_n)$ converging to 0 there exist $(x_{i,n}, \xi_{i,n}, A_{i,n}), i = 1, 2$, such that for every $n = 1, 2, ...,$ $(\xi_{1,n}, A_{1,n}) \in J^{2,-} f(x_{1,n})$, $(\xi_{2,n}, A_{2,n}) \in J^{2,-} g(x_{2,n})$, $d(x_{i,n}, x_n) < \varepsilon_n$, for $i = 1, 2$, $|f(x_{1,n}) - f(x_n)| < \varepsilon_n$, $|g(x_{2,n}) - g(x_n)| < \varepsilon_n$,

$$\|L_{x_{1,n} x_n}(\xi_{1,n}) + L_{x_{2,n} x_n}(\xi_{2,n}) - \xi_n\| < \varepsilon_n, \tag{3.4}$$

and

$$L_{x_{1,n} x_n}(A_{1,n}) + L_{x_{2,n} x_n}(A_{2,n}) \geq A_n - \varepsilon_n I. \tag{3.5}$$

Set

$$\mu_n = \min\{\|B_1\| + \|B_2\| : \ B_1 \in J^{2,-}f(x_{1,n}, \xi_{1,n}), B_2 \in J^{2,-}g(x_{2,n}, \xi_{2,n}), \\ L_{x_{1,n}x_n}(B_1) + L_{x_{2,n}x_n}(B_2) \geq A_n - \varepsilon_n I\}. \tag{3.6}$$

Without loss of generality we can suppose that $\|A_{1,n}\| + \|A_{2,n}\| = \mu_n$ and $A_{1,n} \in J^{2,e}f(x_{1,n}, \xi_{1,n})$, $A_{2,n} \in J^{2,e}g(x_{2,n}, \xi_{2,n})$ for all $n = 1, 2, ....$

Obviously, $(x_{1,n}, f(x_{1,n})) \to (x, f(x))$, and $(x_{2,n}, g(x_{1,n})) \to (x, g(x))$.

Now we define the sequence $(r_n)$ as follows:

$$r_n = \|A_{1,n}\| + \|A_{2,n}\| + \|\xi_{1,n}\| + \|\xi_{2,n}\|.$$

If $(r_n)$ is bounded, then $\|L_{x_{i,n}x}(\xi_{i,n})\|$ $i = 1, 2$ is bounded in $T_xM^*$, thus it has a convergent subsequence. Without relabeling we assume that $L_{x_{i,n}x}(\xi_{i,n})$ tends to $\eta_i$, for $i = 1, 2$. Hence for every $C^\infty$-vector field $V$ on a neighborhood of $x \in M$,

$$\langle \xi_{i,n}, V(x_{i,n})\rangle = \langle L_{x_{i,n}x}(\xi_{i,n}), L_{x_{i,n}x}(V(x_{i,n}))\rangle \to \langle \eta_i, V(x)\rangle,$$

which means $\xi_{i,n}$ converges to $\eta_i$ for $i = 1, 2$. Similarly one can prove that $A_{i,n}$ has a convergent subsequence to an element $B_i$, $i = 1, 2$. We shall prove that $(\eta_1, B_1) + (\eta_2, B_2) = (\xi, A)$. By (3.4),

$$\|L_{x_{1,n}x_n}(\xi_{1,n}) + L_{x_{2,n}x_n}(\xi_{2,n}) - L_{xx_n}(\xi)\| \\ \leq \|L_{x_{1,n}x_n}(\xi_{1,n}) + L_{x_{2,n}x_n}(\xi_{2,n}) - \xi_n\| + \|L_{xx_n}(\xi) - \xi_n\| \to 0.$$

Moreover, changing $A_{i,n}$ into

$$A_{i,n} - 1/2(L_{x_nx_{i,n}}(L_{x_{1,n}x_n}(A_{1,n}) + L_{x_{2,n}x_n}(A_{2,n}) - (A_n - \varepsilon_n I))), \text{ for } i = 1, 2,$$

we obtain (3.5) in the equality form and conclude $B_1 + B_2 = A$.

It remains to obtain a contradiction when $(r_n) \to \infty$. In this case we define $\omega_{i,n} = r_n^{-1}\xi_{i,n}$ and $C_{i,n} = r_n^{-1}A_{i,n}$, $i = 1, 2$. We can assume that these sequences converge to some $\omega_i$ and $C_i$ respectively, for $i = 1, 2$. Hence $(\omega_1, C_1) \in J^{2,\infty}f(x)$, $(\omega_2, C_2) \in J^{2,\infty}g(x)$ and (3.5), (3.4) imply that $\omega_1 + \omega_2 = 0$, $C_1 + C_2 \geq 0$. By assumption, $\omega_1 = \omega_2 = 0$, $C_1 \geq 0$, $C_2 \geq 0$, and $\|C_1\| + \|C_2\| = 1$. Set

$$D_n = L_{x_{1,n}x_n}(A_{1,n}) + L_{x_{2,n}x_n}(A_{2,n}) - (A_n - \varepsilon_n I) \geq 0,$$

and

$$A'_{i,n} = A_{i,n} - \beta_i L_{x_nx_{i,n}}(D_n) = (1 - \beta_i)A_{i,n} + \beta_i(A_{i,n} - L_{x_nx_{i,n}}(D_n)),$$

where $\beta_i = \|C_i\|$, for $i = 1, 2$. Note that $A_{1,n} \in J^{2,-}f(x_{1,n}, \xi_{1,n})$, thus Proposition 1.1 implies that for any $\varepsilon > 0$ the function

$$v \mapsto f \circ \exp_x(v) - f \circ \exp_x(0) - \langle \xi_{1,n}, v\rangle_x - \frac{1}{2}\langle A_{1,n}v, v\rangle_x + \varepsilon\|v\|^2,$$

has a local minimum at $0_x$. Hence

$$0 \leq f \circ \exp_x(v) - f \circ \exp_x(0) - \langle \xi_{1,n}, v\rangle_x - \frac{1}{2}\langle A_{1,n}v, v\rangle_x + \varepsilon\|v\|^2 \\ \leq f \circ \exp_x(v) - f \circ \exp_x(0) - \langle \xi_{1,n}, v\rangle_x - \frac{1}{2}\langle A'_{1,n}v, v\rangle_x + \varepsilon\|v\|^2,$$

which means $A'_{1,n} \in J^{2,-}f(x_{1,n}, \xi_{1,n})$. Similarly, $A'_{2,n} \in J^{2,-}g(x_{2,n}, \xi_{2,n})$. Moreover, $L_{x_{1,n}x_n}(A'_{1,n}) + L_{x_{2,n}x_n}(A'_{2,n}) = A_n - \varepsilon_n I$. Therefore we will arrive at a contradiction if we prove that $\|A'_{1,n}\| + \|A'_{2,n}\| < \mu_n$.

We define the sequence $\alpha_n$ converging to zero as follows,

$$\alpha_n = \|r_n^{-1}(A_n - \varepsilon_n I)\| + \|r_n^{-1} L_{x_n x_{2,n}}(L_{x_{1,n}x_n}(A_{1,n})) - L_{xx_{2,n}}(C_1)\| \\ + \|r_n^{-1} L_{x_n x_{1,n}}(L_{x_{2,n}x_n}(A_{2,n})) - L_{xx_{1,n}}(C_2)\|.$$

Since $C_2 \geq 0$ it follows that for each $v$ in the unite sphere $S$ of $T_{x_{1,n}}M^*$,

$$\begin{aligned} &\langle r_n^{-1} A'_{1,n} v, v\rangle \\ &= (1-\beta_1)\langle r_n^{-1} A_{1,n} v, v\rangle + \beta_1(\langle r_n^{-1}(A_{1,n} - L_{x_n x_{1,n}}(D_n))v, v\rangle) \\ &\leq (1-\beta_1)\langle r_n^{-1} A_{1,n} v, v\rangle + \beta_1(\langle r_n^{-1}(A_{1,n} - L_{x_n x_{1,n}}(D_n)) + L_{xx_{1,n}}(C_2)v, v\rangle). \end{aligned}$$

Let us extract a subsequence of $r_n^{-1}A_{1,n}$ (without relabeling) such that

$$\|r_n^{-1} A_{1,n}\| \leq \beta_1 + \frac{1}{n}.$$

Thus it can be deduced that

$$|\langle r_n^{-1} A'_{1,n} v, v\rangle| \leq (1-\beta_1)(\beta_1 + \frac{1}{n}) + \beta_1 \alpha_n.$$

On the other hand $r_n \sim \mu_n$ and

$$\limsup \mu_n^{-1}(\|A'_{1,n}\| + \|A'_{2,n}\|) < 1,$$

that means $\|A'_{1,n}\| + \|A'_{2,n}\| < \mu_n$ for $n$ large enough, which is a contradiction. □

In the following theorem using a qualification assumption based on a second order singular subjet and cosubjet, a second order chain rule is proved.

**Theorem 3.7.** *Let $M$ be a complete Riemannian manifold with positive sectional curvature and $F : M \to \mathbb{R}^m$ be a map of class $C^1$ near $x \in M$. Suppose that $g : \mathbb{R}^m \to \mathbb{R}$ is lower semicontinuous near $y = F(x)$ and finite at $y$. Set $f = g \circ F$ and assume that the following condition is satisfied:*

$$(y^*, B) \in J^{2,\infty}g(y),\ (0, C) \in D^2_\infty F(x, y^*), \\ dF(x)^* \circ B \circ\ dF(x) + C \geq 0 \Rightarrow\ y^* = 0,\ B = 0.$$

*Then for any $(\xi, A) \in \bar{J}^{2,-}f(x)$, there exist $(y^*, B) \in \bar{J}^{2,-}g(y)$ and $C \in \mathcal{L}^2_s(T_x M)$ such that $(\xi, C) \in \overline{D}^2 F(x, y^*)$ and*

$$A = dF(x)^* \circ B \circ dF(x) + C.$$

*Proof.* Assume that $(\xi, A) \in \bar{J}^{2,-}f(x)$. Hence there exist $x_n \in M$, $(\xi_n, A_n) \in J^{2,-}f(x_n)$ such that $(x_n, f(x_n), \xi_n, A_n) \to (x, f(x), \xi, A)$.

Theorem 2.7 implies for a given positive sequence $(\varepsilon_n)$ converging to 0, there are $z_n \in M$, $\gamma_n \in T_{z_n}M^*$, $r_n, \zeta_n, \eta_n \in \mathbb{R}^m$ and $B_n \in \mathcal{L}^2_s(\mathbb{R}^m), C_n \in \mathcal{L}^2_s(T_{z_n}M)$ such that $d(z_n, x_n) < \varepsilon_n$, $\|r_n - y_n\| < \varepsilon_n$ for $y_n = F(x_n)$, $\|L_{z_n x_n}\gamma_n - \xi_n\| < \varepsilon_n$, $\|\zeta_n - \eta_n\| < \varepsilon_n$ and

$$(\gamma_n, C_n) \in J^{2,-}(\eta_n \circ F)(z_n),\ (\zeta_n, B_n) \in J^{2,-}g(r_n), \tag{3.7}$$

$$C_n + dF(z_n)^* \circ B_n \circ dF(z_n) \geq L_{x_n z_n} A_n - \varepsilon_n I. \tag{3.8}$$

Set

$\mu_n = \min\{\|B\| + \|C\| : \ B \times C \in \mathcal{L}_s^2(\mathbb{R}^m) \times \mathcal{L}_s^2(T_{z_n}M),\ (3.7) \text{ and } (3.8) \text{ hold } \}.$

Without loss of generality we can suppose that $\|B_n\| + \|C_n\| = \mu_n$, for all $n = 1, 2, \ldots$.

Now we define the sequence $(w_n)$ as follows:

$$w_n = \|B_n\| + \|C_n\| + \|\eta_n\|.$$

If $(w_n)$ is bounded, then we conclude the proof. In the case that $(w_n) \to \infty$, introducing $C_n' = L_{x_n z_n}(A_n - \varepsilon_n I) - dF(z_n)^* \circ B_n \circ dF(z_n)$ leads to a contradiction along the same lines as the proof of the previous theorem. □

**Acknowledgment**

The authors would like to thank the anonymous referee for his/her useful comments that helped to improve the presentation of the paper.

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Received: March 18, 2011.
Revised: August 7, 2011.
Accepted: October 19, 2011.